

# Spinning strings and AdS/CFT duality

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## Abstract

We review a special class of semiclassical string states in  $AdS_5 \times S^5$  that have a regular expansion of their energy in integer powers of the ratio of the square of string tension ('t Hooft coupling) and the square of large angular momentum in  $S^5$ . They allow one to quantitatively check the AdS/CFT duality in non-supersymmetric sector of states and also help to uncover the role of integrable structures on the two sides of the string theory – gauge theory duality.

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# 1 Introduction

Better understanding of the duality between type IIB superstring theory in  $AdS_5 \times S^5$  and planar limit of  $\mathcal{N} = 4$  supersymmetric Yang-Mills theory [1] and extending it to less supersymmetric cases may allow us to find simple string-theoretic descriptions of various dynamical aspects of gauge theories, from high-energy scattering to confinement. This AdS/CFT duality is usually viewed as an example of strong coupling – weak coupling duality: while the large  $N$  perturbative expansion in SYM theory assumes that the 't Hooft coupling  $\lambda = g_{\text{YM}}^2 N$  is small, the string perturbative (inverse tension) expansion applies for  $\sqrt{\lambda} = \frac{R^2}{\alpha'} \gg 1$ . In general, observables (like scaling dimensions, correlation functions, finite temperature free energy, etc.) depend on  $\lambda$  through non-trivial functions  $f(\lambda)$  of the couplings, with the perturbative SYM and string theories describing opposite asymptotic regions. For special “protected” BPS observables the dependence on  $\lambda$  may become trivial due to supersymmetry and then can be directly reproduced on the two sides of the duality [2].

Checking duality beyond the BPS cases remains a challenge. The Green-Schwarz superstring action in  $AdS_5 \times S^5$  appears to have a complicated structure [3], so finding, e.g., its full spectrum exactly in  $\lambda$  seems hard at present. One may hope to by-pass the computation of non-trivial functions  $f(\lambda)$  by considering special limits involving other parameters or quantum numbers besides  $\lambda$ . A specific progress can be made by concentrating on a particular (and basic) class of observables which should be related according to AdS/CFT: a spectrum of energies of single-string states (in global  $AdS_5 \times S^5$  coordinates) and scaling dimensions of the corresponding single-trace gauge-invariant local operators in SYM theory. These may carry quantum numbers like  $SO(2, 4) \times SO(6)$  spins of string states or powers of scalar fields and covariant derivatives in the SYM operators. A remarkable recent development initiated in [4] (which in turn was inspired by [5, 6]) is based on the idea that for a special subset of string/SYM states parametrized by large quantum numbers [7, 8] there may be new interesting limits in which certain quantum corrections may be suppressed. One may then be able to check the AdS/CFT for such special non-supersymmetric states by comparing the corresponding string energies to the perturbative gauge-theory scaling dimensions.

In the BMN case [4] (see [9] for reviews) one concentrates on a particular “semi-classical” [8] sector of near-BPS states represented by small closed strings with center of mass moving along a large circle of  $S^5$  with angular momentum  $J \gg 1$ . The SYM operators are of the type  $\text{tr}(Z^J \dots)$ ,  $Z = \Phi_5 + i\Phi_6$  where  $\Phi_M$  are  $SO(6)$  scalars and dots stand for a small number of other SYM fields or covariant derivatives. By considering the limit  $J \rightarrow \infty$ ,  $\frac{\lambda}{J^2} = \text{fixed}$  one is able to establish a precise correspondence between the energies of the string states and scaling dimensions of the corresponding SYM operators [4, 10] (for a complete list of references see [9]). The reason why this is possible can be understood by interpreting this sector of states as “semiclassical” states [8] corresponding to quadratic fluctuations near point-like string running along a geodesic in  $S^5$  with angular velocity  $w = \frac{J}{\sqrt{\lambda}}$ . One is then able to argue [11, 12]

that higher than 1-loop string sigma model corrections to the leading (“quadratic” or “plane-wave”) string energies are suppressed in the limit  $J \rightarrow \infty$ ,  $w=\text{fixed}$ .

One may hope to apply a similar reasoning to other, far from BPS, semiclassical sectors of string states. For example, considering a string rotating with large spin  $S$  in  $AdS_5$  one discovers [8] a new *qualitative* test of AdS/CFT: the agreement between the dependence of the string energy  $E$  on large spin  $S$  and the spin dependence of the anomalous dimension of twist 2 gauge-invariant SYM operators:  $E = S + f(\lambda) \ln S + \dots$ . Here  $f(\lambda) = b_0 \sqrt{\lambda} + b_1 + \frac{b_2}{\sqrt{\lambda}} + \dots$  on perturbative string side and  $f(\lambda) = a_1 \lambda + a_2 \lambda^2 + \dots$  on perturbative gauge theory side [8, 11, 13]. According to the AdS/CFT duality the two expansions must represent different asymptotics of the same function. Checking this in a precise manner is obviously hard since that would require first finding all terms in the respective perturbative series and then resumming them.

For other semiclassical string states one might expect to find similar “interpolation in  $\lambda$ ” pattern, precluding the direct *quantitative* comparison to perturbative SYM theory. Remarkably, as was noticed in [14, 15], there are exceptions: for certain multispin string states (with at least one large  $S^5$  spin component  $J$ ) the classical energy has a *regular* expansion in  $\frac{\lambda}{J^2}$  while quantum superstring sigma model corrections are suppressed in the limit  $J \rightarrow \infty$ ,  $\frac{\lambda}{J^2}=\text{fixed}$ . It was proposed [14] that for such states one can carry out the precise test of the AdS/CFT duality in a non-BPS sector by comparing the  $\frac{\lambda}{J^2} \ll 1$  expansion of the *classical* string energy with the corresponding *quantum* anomalous dimensions in perturbative SYM theory.

This was indeed successfully accomplished in a series of recent papers [16, 17, 18, 19, 20, 21]. The main technical problem – how to find eigenvalues of anomalous dimension matrix for “long” (large  $J$ ) scalar operators – was solved (at the one-loop level) using the interpretation of the anomalous dimension matrix as an integrable spin-chain Hamiltonian [22, 23].<sup>†</sup> This allowed one to find the one-loop anomalous

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<sup>†</sup>Integrable spin chain connection was uncovered and extensively studied previously in QCD context. In particular, the Regge asymptotics of scattering amplitudes was described by evolution equations that were related to the  $SL(2, C)$  Heisenberg spin chain [24]. More importantly for the present discussion, the one-loop anomalous dimensions of certain (quasipartonic) composite operators were related to the energies of the  $SL(2, R)$  XXX Heisenberg spin chain [25, 26]. Similar relations hold in other asymptotically free gauge theories, in particular, supersymmetric theories [27, 28]. The role of conformal symmetry in QCD and these integrability relations were reviewed in [29]. More recent work relating integrability of light-cone QCD operators to gauge/string duality appeared in [30, 31, 32]. In  $\mathcal{N}=4$  SYM theory viewed as a particular gauge theory with adjoint matter the above QCD-inspired work implies that the (1-loop, large  $N$ ) anomalous dimension matrix for the minimal-twist operators (like  $\text{tr}(\Phi D^S \Phi) + \dots$ ,  $D = D_0 + D_3$ ) should be the same as the Hamiltonian of the  $SL(2, R)$  XXX spin chain. Independently, it was observed in [22] that the one-loop planar anomalous dimension matrix in the pure-scalar sector of operators  $\text{tr}(\Phi_{M_1} \dots \Phi_{M_J})$  can be interpreted as a Hamiltonian of an integrable  $SO(6)$  spin chain. Ref.[23] generalized these facts to all superconformal operators to claim that the complete one-loop planar dilatation operator of  $\mathcal{N}=4$  SYM is equivalent to a Hamiltonian of an integrable  $SU(2, 2|4)$  (super) spin chain. More recent work [33] addressed the same problem using the original (operators on light-cone) QCD approach, i.e. considering the subsector of supermultiplets of quasipartonic operators ( $\text{tr}(D^{s_1} \Phi_{M_1} \dots D^{s_n} \Phi_{M_n}) + \dots$ ,

dimensions by applying the Bethe ansatz [34] techniques. The leading order  $\frac{\lambda}{J}$  terms in the energies of particular string solutions were then reproduced as one-loop anomalous dimensions on the SYM side by choosing particular Bethe root distributions in the “thermodynamic” limit of “long” ( $J \rightarrow \infty$ ) operators. There is some evidence [19] that the correspondence extends, as one of course expects, to the next  $\frac{\lambda^2}{J^3}$  order, but checking this explicitly and going beyond the 2-loop level remains an important open problem.

Our aim here will be to review a class of such classical string solutions in  $AdS_5 \times S^5$  [11, 14, 15, 17, 18, 35] whose energy  $E$  has a regular expansion in integer powers of  $\lambda$  (i.e. the square of the effective string tension) divided over the square of the total  $S^5$  spin  $J$ , and for which quantum sigma model corrections to  $E$  should be suppressed in the  $J \rightarrow \infty$  limit.

Let us first make some general comments on the structure of this semiclassical expansion for the string energy. The form of a classical solution cannot depend on the value of the string tension, i.e. on  $\sqrt{\lambda}$ , which appears as a factor in front the string action  $I = \frac{\sqrt{\lambda}}{4\pi} \int d^2\xi G_{\mu\nu}(x) \partial_a X^\mu \partial^a X^\nu + \dots$ . Thus the classical energy can be written as  $E = \sqrt{\lambda} \mathcal{E}(w)$ , where  $w$  stands for all constant parameters that enter the classical solution. These parameters should be fixed in the standard sigma model loop ( $\alpha' \sim \frac{1}{\sqrt{\lambda}}$ ) expansion. The classical values of the integrals of motion like the  $S^5$  and  $AdS_5$  angular momentum components are also proportional to the string tension, e.g.,  $J = \sqrt{\lambda} \mathcal{J}(w)$  (they take integer values in the full quantum theory). Expressed in terms of these integrals the classical energy is  $E = \sqrt{\lambda} \mathcal{E}(\mathcal{J}) = \sqrt{\lambda} \mathcal{E}(\frac{J}{\sqrt{\lambda}})$ .

In the limit of large values of semiclassical parameters and the corresponding quantum charges the classical energy of a string solution in any  $AdS_p \times S^m$  space goes as *linear* function of  $J$ , i.e.  $E = J + \dots$ . This linear behaviour [36] (seen explicitly on examples of particular solutions [37, 8, 11, 38, 39]) is different from the flat-space Regge one  $E \sim \sqrt{J}$  and is a consequence of the constant curvature of  $AdS$  space. This is consistent with the AdS/CFT duality: one expects that the large  $J$  expression for the full dimension of the corresponding gauge-theory operator should start with its canonical dimension.

We would like to identify a class of special classical string solutions in  $AdS_5 \times S^5$  whose energy has a particular dependence on conserved charges that allows for a direct comparison with anomalous dimensions on the perturbative SYM side. While such extended string solutions turn out to have *several* conserved global charges, here for notational simplicity we shall keep track of just one of them – total  $S^5$  angular momentum  $J = \sqrt{\lambda} \mathcal{J}$ . For the solutions we will be interested in the classical energy  


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etc.) with the conclusion that in this case the one-loop dilatation operator coincides with the Hamiltonian of  $SL(2|4)$  spin chain. The relation between the approaches of [23] and [33] and also whether the  $SL(2)$  integrability in the twist 2 sector may be somehow related by supersymmetry to  $SO(6)$  integrability in the pure-scalar sector seems worth clarifying further.

$E = \sqrt{\lambda} \mathcal{E}$  should have the following expansion in large classical parameter  $\mathcal{J} \gg 1$

$$\mathcal{E} = \mathcal{J} \left( 1 + \frac{c_1}{\mathcal{J}^2} + \frac{c_2}{\mathcal{J}^4} + \dots \right), \quad (1.1)$$

i.e.  $\frac{\mathcal{E}}{\mathcal{J}}$  should have an expansion in *even* inverse powers of  $\mathcal{J}$ . The coefficients  $c_i$  may be functions of ratios of conserved charges that are finite in the large-charge limit. Equivalently, for  $\frac{1}{\mathcal{J}} = \frac{\sqrt{\lambda}}{J} \ll 1$

$$E = J \left( 1 + \frac{c_1 \lambda}{J^2} + \frac{c_2 \lambda^2}{J^4} + \dots \right) = J + \frac{c_1 \lambda}{J} + \frac{c_2 \lambda^2}{J^3} + \dots, \quad (1.2)$$

which formally looks like an expansion in positive integer powers of  $\lambda$ . Rotating string solutions with this property were indeed found in [11, 14, 17, 18, 35] and will be reviewed below.

Furthermore, let us assume that in such cases the string sigma model loop corrections to the energy which in general can be computed in the standard inverse tension expansion

$$E_{\text{tot}} = \sqrt{\lambda} \mathcal{E}(\mathcal{J}) + \mathcal{E}_1(\mathcal{J}) + \frac{1}{\sqrt{\lambda}} \mathcal{E}_2(\mathcal{J}) + \frac{1}{(\sqrt{\lambda})^2} \mathcal{E}_3(\mathcal{J}) + \dots = E + \sum_{n=1}^{\infty} E_n \quad (1.3)$$

should have the following specific form of their expansion in  $\mathcal{J} \gg 1$

$$\mathcal{E}_n(\mathcal{J}) = \frac{d_{n1}}{\mathcal{J}^{n+1}} + \frac{d_{n2}}{\mathcal{J}^{n+3}} + \dots, \quad n = 1, 2, \dots \quad (1.4)$$

This behaviour was verified in [15] for  $n = 1$  on a particular example of a solution [14] satisfying (1.2). Eq.(1.4) implies that the  $n$ -loop term in the quantum-corrected energy (1.3) will be given, for  $\frac{\lambda}{J^2} \ll 1$ , by

$$E_n = \frac{1}{(\sqrt{\lambda})^{n-1}} \mathcal{E}_n(\mathcal{J}) = \frac{d_{n1} \lambda}{J^{n+1}} + \frac{d_{n2} \lambda^2}{J^{n+3}} + \dots \quad (1.5)$$

In general, the energy  $E_{\text{tot}} = E_{\text{tot}}(J, \lambda)$  should be some function of  $J$  and the string tension but if the above assumptions (1.2) and (1.5) are true, it will be given by the following *double expansion* in  $\frac{\lambda}{J^2}$  and  $\frac{1}{J}$ :

$$E_{\text{tot}} = J \left[ 1 + \sum_{k=1}^{\infty} \left( \frac{\lambda}{J^2} \right)^k \left( c_k + \sum_{n=1}^{\infty} \frac{d_{nk}}{J^n} \right) \right]. \quad (1.6)$$

Then if we first take the limit of  $J \gg 1$  for fixed  $\frac{\lambda}{J^2}$ , all quantum sigma model corrections will be suppressed and the full energy  $E_{\text{tot}}$  will be given just by its *classical* part  $E$  (1.2).

In the BMN case [4], where one expands [8, 11] near a point-like BPS string state, the ground-state energy is not renormalized, i.e.  $E_{\text{tot}} = E = J$ , but the double

expansion similar to the one for  $E_{\text{tot}}/J$  in (1.6), namely,  $E_{\text{fluct}} = J + \sum_{k=0}^{\infty} (\frac{\lambda}{J^2})^k (h_k + \sum_{n=1}^{\infty} \frac{f_{nk}}{J^n})$ , applies to energies of string fluctuations near the geodesic, i.e. to energies of excited string modes [11, 12, 40, 41]. In the limit  $J \rightarrow \infty$  their energies are then determined by the quadratic (“one-loop” or “plane-wave”) approximation.

The general conditions for the validity of the expansions (1.2) and (1.5) remain to be clarified (some observations made in recent papers [42, 43] relating large  $\mathcal{J}$  limit to an ultra-relativistic limit may turn out to be useful for that). In particular, the “regularity” of the expansion of the energy in  $\lambda$  (1.2) may apply not only to multi-spin rotating but also to  $S^5$  pulsating [39, 21] solutions.

In order to test the AdS/CFT duality one should reproduce the same expression for the (quantum-corrected)  $AdS_5$  string energy  $E_{\text{tot}}(\lambda, J)$  (1.6) as the exact scaling dimension  $\Delta(\lambda, J)$  of the corresponding SYM operator, i.e. as a particular eigenvalue of the dilatation operator having the same global charges (i.e. belonging to the same  $SO(2, 4) \times SO(6)$  representation as the string state). Given that (1.6) looks like an expansion in the ‘t Hooft coupling  $\lambda$  it is natural to expect that the perturbative ( $\lambda \ll 1$ ) expansion for  $\Delta(\lambda, J)$  can be organized in the following way

$$\Delta(\lambda, J) = J + \sum_{k=1}^{\infty} q_k(J) \lambda^k, \quad (1.7)$$

where the functions  $q_k(J)$  should have the following form for  $J \gg 1$

$$q_k(J) = \frac{1}{J^{2k-1}} (a_k + \frac{a_{k1}}{J} + \dots). \quad (1.8)$$

Assuming that this is indeed the case and taking the  $J \rightarrow \infty$  limit, one could then directly compare the classical part of the energy in (1.6) expanded in  $\frac{\lambda}{J^2}$  with the sum of the leading ( $J \gg 1$ ) terms at each order of expansion of  $\Delta$  in powers of  $\lambda$ . The AdS/CFT implies then that the two expressions should coincide, i.e. that  $c_k = a_k$ . The *classical* string energy should thus represent the leading  $J \rightarrow \infty$  term in the *quantum* SYM scaling dimension.

In particular, the coefficient  $c_1$  of the first subleading (order  $\lambda$ ) term in the classical string energy (1.2) should match the coefficient  $a_1$  in the one-loop SYM term in (1.7),(1.8). This was indeed verified on specific examples in [16, 17, 18, 19, 21]. There is also a numerical evidence [19] that this matching extends to the  $\lambda^2$  term, i.e.  $c_2 = a_2$ .

The  $J \rightarrow \infty$  behavior (1.8) of the one-loop correction to the anomalous dimensions was checked using the spin chain relation and the Bethe ansatz for particular large R-charge or large spin operators [16, 19, 21]. The general proof of (1.8) which should follow from a higher-loop structure of the dilatation operator [45, 46] and should be heavily based on the superconformal symmetry of the  $\mathcal{N} = 4$  SYM theory remains to be given.

Let us now summarize the contents of the following sections. In section 2 we shall write down the bosonic part of the superstring action in  $AdS_5 \times S^5$  and the



corresponding integrals of motion as a preparation for a discussion of classical finite energy closed string solutions which carry several  $SO(2, 4) \times SO(6)$  spin components. In section 3 we shall consider the special case of  $SO(6)$  invariant sigma model (embedded into string theory by adding time direction from  $AdS_5$ ) and briefly review its integrability (local and non-local conserved charges, etc.).

Then in section 4 we shall concentrate on a particular class of semiclassical string states rotating in  $S^5$  with three angular momenta  $J_i$  and show that for the rotating string ansatz the  $R_t \times S^5$  sigma model reduces to a well-known one-dimensional integrable system – Neumann-Rosochatius (NR) system. Its special case is the  $n = 3$  Neumann system describing an oscillator on a 2-sphere. This relation allows one to classify the corresponding rotating string solutions, which, as in flat space, can be of folded or circular type.

In section 5 we shall study a simple special class of circular rotating string solutions on  $S^5$  whose energy has a regular large-spin expansion as in (1.2). We shall also determine (in section 5.3) the spectrum of quadratic fluctuations near these circular solutions pointing out some analogy with the point-like (BMN) case. In section 5.4 we shall consider the one-loop string sigma model correction to the energy of a particular solution (with two equal spins); this one-loop correction indeed turns out to be suppressed in the large spin limit, in agreement with (1.6).

The discussion of sections 4 and 5 will be generalized in section 6 to the case of states represented by semiclassical strings rotating in both  $S^5$  and  $AdS_5$  and thus carrying 3+2 spin quantum numbers. They are again described by a generalised NR integrable system. While the energy of strings rotating only in  $AdS_5$  is non-analytic in  $\lambda$  (section 6.1), the expansion (1.2) is true for circular strings having large  $S^5$  spin components.

Similar conclusions apply to other multi-spin solutions of the NR system representing folded and circular strings with more complicated (“inhomogeneous”) dependence on the string coordinate  $\sigma$ . In particular, we consider a class of two-spin solutions for which the Neumann system degenerates to a sine-Gordon one and, as a result, the solutions are expressed in terms of the elliptic functions (section 7). The classical energy can then be found as a solution of the two parametric equations involving elliptic integrals and has again a regular expansion as in (1.2).

Section 8 will contain a summary of some open problems and possible generalizations, including a brief discussion of pulsating string solutions.

## 2 Closed superstrings in $AdS_5 \times S^5$ : classical solutions

Superstrings in  $AdS_5 \times S^5$  can be described by a Green-Schwarz action [3] which defines a consistent perturbation theory near each semiclassical string configuration, e.g., point-like massless geodesic in a light-cone type gauge as in [47, 11] or extended string configurations as in [48, 11, 14, 15].



The bosonic part of the action in the conformal gauge is the sum of the two coset-space sigma models ( $AdS_5 = SO(2, 4)/SO(1, 4)$  and  $S^5 = SO(6)/SO(5)$  ones)

$$I = -\frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma [G_{mn}^{(AdS_5)}(x) \partial_a x^m \partial^a x^n + G_{pq}^{(S^5)}(y) \partial_a y^p \partial^a y^q] . \quad (2.1)$$

The effective string tension  $T_{\text{eff}} = \frac{\sqrt{\lambda}}{2\pi} = \frac{R^2}{2\pi\alpha'}$  is related to the ‘t Hooft coupling  $\lambda = g_{\text{YM}}^2 N$  on the SYM side of the string/gauge theory duality [1]. The  $AdS_5$  and  $S^5$  parts of the action are “coupled” at the classical level through the conformal gauge constraints.

The classical conformal invariance of this sigma model is preserved at the quantum level after addition of fermions with coupling to the metric and R-R 5-form background [3]. There are quadratic and quartic fermionic terms in the action (in a particular gauge). The quadratic part of the fermionic Lagrangian can be written as (see, e.g., [3, 48, 11])

$$L_F = i(\eta^{ab} \delta^{IJ} - \epsilon^{ab} s^{IJ}) \bar{\vartheta}^I \varrho_a D_b \vartheta^J , \quad \varrho_a \equiv \Gamma_A e_a^A , \quad (2.2)$$

where  $I, J = 1, 2$ ,  $s^{IJ} = \text{diag}(1, -1)$ , and  $\varrho_a \equiv \Gamma_A E_\mu^A \partial_a X^\mu$  are the projections of the 10-d Dirac matrices. Here  $X^\alpha$  are the string coordinates (given functions of  $\tau$  and  $\sigma$  for a particular classical solution) corresponding to the  $AdS_5$  ( $\mu = 0, 1, 2, 3, 4$ ) and  $S^5$  ( $\mu = 5, 6, 7, 8, 9$ ) factors. The covariant derivative  $D_a$  can be put into the following form

$$D_a \vartheta^I = (\delta^{IJ} D_a - \frac{i}{2} \epsilon^{IJ} \Gamma_* \varrho_a) \vartheta^J , \quad \Gamma_* \equiv i\Gamma_{01234} , \quad \Gamma_*^2 = 1 , \quad (2.3)$$

where  $D_a = \partial_a + \frac{1}{4} \omega_a^{AB} \Gamma_{AB}$ ,  $\omega_a^{AB} \equiv \partial_a X^\alpha \omega_\alpha^{AB}$  and the “mass term” originates from the R-R 5-form coupling.

Here we will be interested mostly in the classical bosonic finite-energy solutions for closed strings in  $AdS_5 \times S^5$  space and ignore the fermions. To study these bosonic solutions it is useful to rewrite the action (2.1) in the form

$$I = \sqrt{\lambda} \int d\tau \int_0^{2\pi} \frac{d\sigma}{2\pi} (L_{AdS} + L_S) , \quad (2.4)$$

where

$$L_{AdS} = -\frac{1}{2} \eta^{PQ} \partial_a Y_P \partial^a Y_Q + \frac{1}{2} \tilde{\Lambda} (\eta^{PQ} Y_P Y_Q + 1) , \quad (2.5)$$

$$L_S = -\frac{1}{2} \partial_a X_M \partial^a X_M + \frac{1}{2} \Lambda (X_M X_M - 1) . \quad (2.6)$$

We use  $(-+)$  signature on the world sheet and  $X_M$ ,  $M = 1, \dots, 6$  and  $Y_P$ ,  $P = 0, \dots, 5$  are the embedding coordinates of  $R^6$  with the Euclidean metric  $\delta_{MN}$  in  $L_S$  and with  $\eta_{PQ} = (-1, +1, +1, +1, +1, -1)$  in  $L_{AdS}$ , respectively.  $\Lambda$  and  $\tilde{\Lambda}$  are the Lagrange multiplier functions of  $\tau$  and  $\sigma$ . The action (2.4) is to be supplemented with the

usual conformal gauge constraints expressing the vanishing of the total 2-d energy-momentum tensor

$$\eta^{PQ}(\dot{Y}_P \dot{Y}_Q + Y'_P Y'_Q) + \dot{X}_M \dot{X}_M + X'_M X'_M = 0 , \quad (2.7)$$

$$\eta^{PQ} \dot{Y}_P Y'_Q + \dot{X}_M X'_M = 0 , \quad (2.8)$$

where

$$\eta^{PQ} Y_P Y_Q = -1 , \quad X_M X_M = 1 . \quad (2.9)$$

We shall assume that the world sheet is a cylinder, i.e. impose the closed string periodicity conditions

$$Y_P(\sigma + 2\pi) = Y_P(\sigma) , \quad X_M(\sigma + 2\pi) = X_M(\sigma) . \quad (2.10)$$

The classical equations that follow from (2.4) can be written as

$$\partial^a \partial_a Y_P - \tilde{\Lambda} Y_P = 0 , \quad \tilde{\Lambda} = \eta^{PQ} \partial^a Y_P \partial_a Y_Q , \quad \eta^{PQ} Y_P Y_Q = -1 , \quad (2.11)$$

$$\partial^a \partial_a X_M + \Lambda X_M = 0 , \quad \Lambda = \partial^a X_M \partial_a X_M , \quad X_M X_M = 1 . \quad (2.12)$$

The action is invariant under the  $O(2,4)$  and  $O(6)$  global symmetries with the corresponding conserved (on-shell) charges being

$$S_{PQ} = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} (Y_P \dot{Y}_Q - Y_Q \dot{Y}_P) , \quad (2.13)$$

$$J_{MN} = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} (X_M \dot{X}_N - X_N \dot{X}_M) . \quad (2.14)$$

We are interested in finding “spinning” solutions that have non-zero values of these charges. The physical target-space interpretation of the solutions depends on a particular choice of coordinates (that solve (2.9)) in  $AdS_5$  and  $S^5$ . One natural (“global coordinate”) choice is

$$\begin{aligned} Y_1 &\equiv Y_1 + iY_2 = \sinh \rho \sin \theta e^{i\phi_1} , & Y_2 &\equiv Y_3 + iY_4 = \sinh \rho \cos \theta e^{i\phi_2} , \\ Y_0 &\equiv Y_5 + iY_6 = \cosh \rho e^{it} , & X_3 &\equiv X_5 + iX_6 = \cos \gamma e^{i\varphi_3} , \\ X_1 &\equiv X_1 + iX_2 = \sin \gamma \cos \psi e^{i\varphi_1} , & X_2 &\equiv X_3 + iX_4 = \sin \gamma \sin \psi e^{i\varphi_2} . \end{aligned} \quad (2.15)$$

Then there is an obvious choice of the 3+3 Cartan generators of  $SO(2,4) \times SO(6)$  corresponding to the 3+3 linear isometries of the  $AdS_5 \times S^5$  metric

$$(ds^2)_{AdS_5} = d\rho^2 - \cosh^2 \rho dt^2 + \sinh^2 \rho (d\theta^2 + \cos^2 \theta d\phi_1^2 + \sin^2 \theta d\phi_2^2) , \quad (2.16)$$

$$(ds^2)_{S^5} = d\gamma^2 + \cos^2 \gamma d\varphi_3^2 + \sin^2 \gamma (d\psi^2 + \cos^2 \psi d\varphi_1^2 + \sin^2 \psi d\varphi_2^2) , \quad (2.17)$$

i.e. to the translations in  $AdS_5$  time  $t$ , in the two angles  $\phi_a$  and in three angles of  $S^5$   $\varphi_i$ :

$$S_0 \equiv S_{50} \equiv E = \sqrt{\lambda} \mathcal{E} , \quad S_1 \equiv S_{12} = \sqrt{\lambda} \mathcal{S}_1 , \quad S_2 \equiv S_{34} = \sqrt{\lambda} \mathcal{S}_2 , \quad (2.18)$$

$$J_1 \equiv J_{12} = \sqrt{\lambda} \mathcal{J}_1, \quad J_2 \equiv J_{34} = \sqrt{\lambda} \mathcal{J}_2, \quad J_3 \equiv J_{56} = \sqrt{\lambda} \mathcal{J}_3. \quad (2.19)$$

We will be interested in classical solutions that have finite values of the target-space energy  $E$  as well as of the spins  $S_a, J_i$ . For a solution to have a consistent semiclassical approximation, i.e. to correspond to an eigen-state of the Hamiltonian which carries the corresponding quantum numbers (and thus being associated to a particular SYM operator with definite scaling dimension) all other non-Cartan (i.e. non-commuting) components of the symmetry generators (2.13),(2.14) should vanish [11]. These correspond to “highest-weight” states; other members of the multiplet can be obtained by applying rotations to a “highest-weight” solution.

In the above  $R^{2,4}$  embedding representation of  $AdS_5$  the charges of the isometry group  $SO(2,4)$  can be related to the boundary SYM theory conformal group generators as follows ( $\mu, \nu = 0, 1, 2, 3$ ):

$$S_{\mu\nu} = M_{\mu\nu}, \quad S_{\mu 4} = \frac{1}{2}(K_\mu - P_\mu), \quad S_{\mu 5} = \frac{1}{2}(K_\mu + P_\mu), \quad S_{54} = D. \quad (2.20)$$

One can identify the standard spin with  $S_1 = S_{12} = M_{12}$ , the second (conformal) spin with  $S_2 = S_{34} = \frac{1}{2}(K_3 - P_3)$ , and finally the conformal energy with the rotation generator in the 05 plane, i.e. with the global  $AdS_5$  energy,  $E = S_{05} = \frac{1}{2}(K_0 + P_0)$ . \*

Let us first consider point-like string solutions, for which  $Y_P = Y_P(\tau)$ ,  $X_M = X_M(\tau)$ , i.e. massless (cf. (2.7)) geodesics in  $AdS_5 \times S^5$ . As follows from the second-order equations in (2.11),(2.12), in this case  $\Lambda = \text{const}$ ,  $\tilde{\Lambda} = \text{const}$ , i.e.  $Y_P$  and  $X_M$  are given by trigonometric functions. The constraint (2.7) implies that the two frequencies are related:  $\Lambda = -\tilde{\Lambda} > 0$ . Then generic massless geodesic in  $AdS_5 \times S^5$  can be shown to be of the two “irreducible” types (up to a global  $SO(2,4) \times SO(6)$  transformation): (i) massless geodesic that stays entirely within  $AdS_5$ ; (ii) a geodesic that runs along the time direction in  $AdS_5$  and wraps a big circle of  $S^5$ . In the latter case the angular motion in  $S^5$  provides an effective (“Kaluza-Klein”) mass to a particle in  $AdS_5$ , i.e. the corresponding geodesic in  $AdS_5$  is a massive one. Then we can choose the coordinates so that

$$Y_5 + iY_0 = e^{i\kappa\tau}, \quad X_5 + iX_6 = e^{iw\tau}, \quad \kappa = w = \sqrt{\Lambda}, \quad Y_{1,2,3,4} = X_{1,2,3,4} = 0. \quad (2.21)$$

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\*The energy of a string state in global  $AdS_5$  space with boundary  $R \times S^3$  should be equal to the energy of the corresponding SYM state on  $R \times S^3$  (which can be mapped conformally to  $R^4$ ). Through radial quantization this state may be associated to a local operator that creates it. At the same time, the  $AdS_5$  energy or conformal Hamiltonian generates an  $SO(2)$  subgroup of  $SO(2,4)$  while the dilatation operator (whose eigenvalues are scaling dimensions) generates an  $SO(1,1)$  subgroup of the conformal group. Their eigenvalues happen to be the same since the two representations (the unitary one classified by  $SO(4) \times SO(2)$  and the one classified by  $SO(4) \times SO(1,1)$ ) are related by a global  $SO(2,4)$  similarity transformation (see, e.g., [44]). Alternatively, after the Euclidean continuation  $Y_0 \rightarrow iY_{0E}$  (to allow for the mapping from  $R \times S^3$  to  $R^4$ ) one may exchange  $Y_{0E}$  with  $Y_4$  which exchanges the generator  $S_{54} = D$  with  $S_{05} = \frac{1}{2}(P_0 + K_0) = E$ . For all the solutions discussed below  $S_{50} = E \neq 0$  while  $S_{54} = D = 0$ . One could, in principle, apply a similar  $Y_{0E} \rightarrow Y_4$  transformation to string solutions, getting equivalent ones (but more complicated-looking, with radial direction of  $AdS_5$  depending on  $\tau$ ) that would have non-zero values of the  $SO(1,1)$  generator  $S_{54}$ .

Here the only non-vanishing integrals of motion are  $E = J_3 = \sqrt{\lambda} \kappa$ , representing the energy and  $SO(6)$  spin of this BPS state, corresponding to  $\text{tr} Z^{J_3}$  operator in SYM theory. More generally, one may choose any massless geodesic in  $AdS_5$  for which then  $\eta^{PQ} \dot{Y}_P \dot{Y}_Q = -w^2$ . The massless limit  $w \rightarrow 0$  corresponds to  $J_3 \rightarrow 0$ , i.e. the resulting state should be representing a vacuum in string theory or a unit operator in SYM theory [4].

The former case, i.e. the “massless”  $w \rightarrow 0$  limit, is actually subtle: naively, a massless geodesic in  $AdS_5$  does not represent a semiclassical string state in the sense it was defined above. Indeed, for a point-like string moving inside  $AdS_5$  we have  $\eta^{PQ} \dot{Y}_P \dot{Y}_Q = 0$ , i.e.  $\dot{Y}_P = 0$ . Thus in terms of the embedding coordinates the massless geodesic is a straight line

$$Y_P(\tau) = A_P + B_P \tau, \quad \eta^{PQ} B_P B_Q = \eta^{PQ} A_P B_Q = 0, \quad \eta^{PQ} A_P A_Q = -1. \quad (2.22)$$

Then the  $SO(2, 4)$  angular momentum tensor (2.13) is  $S_{PQ} = \sqrt{\lambda} (A_P B_Q - A_Q B_P)$  and can be shown to always have non-vanishing non-Cartan components. Indeed, by applying an  $SO(2, 4)$  rotation we may put the constant vectors  $A_P$  and  $B_P$  in a canonical form:  $A_P = (0, 0, 0, 0, 0, 1)$ ,  $B_P = (p, 0, 0, p, 0, 0)$ , i.e.

$$Y_5 + iY_0 = 1 + ip \tau, \quad Y_3 = p \tau, \quad Y_{1,2,4} = 0. \quad (2.23)$$

Here  $p$  is an arbitrary parameter and  $S_{50} = S_{53} = \sqrt{\lambda} p$ . An alternative choice of the parameters (related to the above one by an  $SO(2, 4)$  rotation with parameter  $u$ ) gives  $Y_5 + iY_0 = \frac{1+u^2}{2u} + i\frac{p}{u}\tau$ ,  $Y_1 + iY_2 = \frac{1-u^2}{2u} + i\frac{p}{u}\tau$ ,  $Y_{3,4} = 0$ .<sup>†</sup> It corresponds to the massless geodesic running parallel to the  $R^{1,3}$  boundary in the Poincare coordinates where  $(ds^2)_{AdS_5} = \frac{1}{z^2}(dx^m dx_m + dz^2)$ :  $x_0 = x_3 = p\tau$ ,  $z = u = \text{const}$  (see also [12]). An expansion near this geodesic is used to define the light-cone gauge in [47], i.e. it should represent a light-cone vacuum state.

Below we would like to study non-trivial ( $\sigma$ -dependent) solitonic solutions of classical closed string equations in  $AdS_5 \times S^5$  that have finite 2-d energy and carry finite space-time energy and spins, i.e. 1+2 plus 3 commuting conserved charges of the  $O(2, 4) \times O(6)$  isometry group. The conformal gauge constraints will then imply a relation between the energy and the spins ( $a = 1, 2$ ;  $i = 1, 2, 3$ )

$$\mathcal{E} = \mathcal{E}(\mathcal{S}_a, \mathcal{J}_i; k_p), \quad \text{i.e.} \quad E = \sqrt{\lambda} \mathcal{E}\left(\frac{S_a}{\sqrt{\lambda}}, \frac{J_i}{\sqrt{\lambda}}; k_p\right), \quad (2.24)$$

where  $k_p$  are “topological” numbers determining particular type (e.g., shape) of the rotating solutions. We will be interested in solutions that have a regular dependence of  $E$  on  $\lambda$  in the large spin limit as in (1.2). A necessary condition for that appears to be to have large total angular momentum in  $S^5$  direction. That applies to both

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<sup>†</sup>In this case in addition to the Cartan components  $E = S_{50} = \sqrt{\lambda} \frac{1+u^2}{2u} p$  and  $S_{12} = -\sqrt{\lambda} \frac{1-u^2}{2u} p$  we also have nonvanishing  $S_{01}$  and  $S_{25}$ .

rotating [14] and oscillating [39] solutions. Note also that rotating solutions in  $S^5$  (but not in  $AdS_5$ ) have a “nearly-BPS” interpretation [42] in the formal  $\lambda \rightarrow 0$  limit.

In general, coset space sigma models are known to be integrable [49, 50]. To make this formal integrability property explicit and useful one needs to specify a class of solutions by choosing a special ansatz for string coordinates. Before discussing particular rotating strings in  $S^5$  and  $AdS_5$  let us first make some general comments on the corresponding classical sigma model and its conserved charges.

### 3 $R_t \times S^5$ sigma model : classical integrability and conserved charges

Let us consider the classical  $S^5$  sigma model embedded into string theory by adding extra time direction  $R_t$ . This may be viewed as a special case of  $AdS_5 \times S^5$  sigma model where the string is placed at the center  $\rho = 0$  of  $AdS_5$  while moving in  $S^5$ .

Introducing  $\xi^\pm = \frac{1}{2}(\tau \pm \sigma)$  and  $\partial_\pm = \partial_\tau \pm \partial_\sigma$  the corresponding equations of motion and conformal gauge constraints can be written as (cf. (2.12))

$$\partial_+ \partial_- X_M + (\partial_+ X_N \partial_- X_N) X_M = 0, \quad X_M X_M = 1, \quad (3.1)$$

$$\partial_+ X_M \partial_+ X_M = (\partial_+ t)^2, \quad \partial_- X_M \partial_- X_M = (\partial_- t)^2, \quad (3.2)$$

where  $t$  satisfies  $\partial_+ \partial_- t = 0$ , which has general solution ( $\kappa = \text{const}$ )

$$t = \kappa \tau + h_+(\tau + \sigma) + h_-(\tau - \sigma). \quad (3.3)$$

The equations (3.1),(3.2) are invariant under 2-d conformal transformations,  $\xi^\pm \rightarrow F_\pm(\xi^\pm)$ , so given a solution  $X_M(\xi^+, \xi^-)$  one can find another one as  $\tilde{X}_M(\xi^+, \xi^-) = X_M(F_+(\xi^+), F_-(\xi^-))$ . One can also use this residual conformal symmetry to make the components of the stress tensor  $\partial_+ X_M \partial_+ X_M$  and  $\partial_- X_M \partial_- X_M$  equal to a constant, or, what is equivalent in the present case, to gauge away  $h_\pm$  in (3.3), putting  $t$  in the form  $t = \kappa \tau$ . When only 3 of  $X_M$ ’s are non-zero (as in the case of the  $O(3)$  invariant sigma model) one can show [49] that (3.1) reduces to the 2-d sine-Gordon equation

$$\partial_+ \partial_- \alpha + \sin \alpha = 0, \quad \cos \alpha = \partial_+ X_M \partial_- X_M. \quad (3.4)$$

Similar reduced systems can be derived also from other  $O(n)$  invariant sigma models [51].\*

The above equations (3.1) admit various special solutions. One is the “flat-space” or “chiral” solution (for which the Lagrange multiplier  $\Lambda$  in (2.12) vanishes):  $X_M = f_M^+(\xi^+)$  or  $X_M = f_M^-(\xi^-)$  for particular values of  $M$ . In contrast to flat-space case, a linear combination of such solutions is no longer a solution, so one may thus say that

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\*A relation to sine-Gordon system appeared previously in the context of strings moving in constant curvature spaces in [52, 53].

(3.1) describes scattering of left-moving and right-moving light-like energy lumps [49]. For chiral  $X_M$  to satisfy the string-theory constraints (3.2) we need to make a special choice of  $h_{\pm}$  in  $t$ .

Let us now review various types of local and non-local conserved currents in this sigma model (see, e.g., [51, 54]). One can define a first-order linear system (Lax pair) [49] whose consistency is equivalent to the equations (3.1):

$$\begin{aligned} \partial_+ R^{(\ell)} &= (1 - \ell^{-1}) j_+ R^{(\ell)} , & \partial_- R^{(\ell)} &= (1 - \ell) j_- R^{(\ell)} , \\ R^{(\ell)} (R^{(\ell)})^T &= (R^{(\ell)})^T R^{(\ell)} = I , \end{aligned} \quad (3.5)$$

where  $R^{(\ell)}$  is an  $so(6)$  matrix and

$$(j_a)_{MN} = 2(X_M \partial_a X_N - X_N \partial_a X_M) . \quad (3.6)$$

One can then construct a new solution from a given one as  $X_M^{(\ell)} = R_{MN}^{(\ell)} X_N$ . Solving (3.5) by the inverse scattering method is subtle due to complications related to the choice of boundary conditions [51, 50] (e.g., on an infinite line, for  $j_a \rightarrow 0$  at spatial infinity the solution  $R^{(\ell)}$  does not have a plane-wave behaviour). Still, (3.5) may be used as a basis for analysing the integrability properties of the sigma model .

One approach is to look at non-abelian (non-commuting) non-local conserved charges related to Yangians [55]. At the same time, it is important also to study an infinite family of commuting local conserved charges whose existence is a manifestation of integrability of the corresponding equations of motion. These may be constructed using the Bäcklund transformation. If  $X_M$  is a given “trial” solution of (3.1), let us define its Bäcklund transform  $X_M^{(\gamma)}$  as another solution satisfying [54, 20]

$$\partial_+ (X_M^{(\gamma)} + X_M) = \frac{1}{2} (1 + \gamma^{-2}) X_N^{(\gamma)} \partial_+ X_N (X_M^{(\gamma)} - X_M) , \quad (3.7)$$

$$\partial_- (X_M^{(\gamma)} - X_M) = -\frac{1}{2} (1 + \gamma^2) X_N^{(\gamma)} \partial_- X_N (X_M^{(\gamma)} + X_M) , \quad (3.8)$$

$$X_M^{(\gamma)} X_M^{(\gamma)} = 1 , \quad X_M X_M = 1 , \quad X_M^{(\gamma)} X_M = \frac{1 - \gamma^2}{1 + \gamma^2} , \quad X_M^{(0)} = X_M .$$

Here  $\gamma$  is a spectral parameter. One can write the solution of the equations (3.8) in expansion in  $\gamma$

$$X_M^{(\gamma)} = X_M + \sum_{k=1}^{\infty} X_{(k)M} \gamma^k , \quad X_{(1)M} = \frac{2\partial_+ X_M}{\sqrt{\partial_+ X_N \partial_+ X_N}} , \dots \quad (3.9)$$

One can define the generating function of *local* commuting conserved *scalar* charges associated with the original solution  $X_M$  by [54, 20]

$$Q(\gamma) = \frac{1}{2} \gamma \int_0^{2\pi} \frac{d\sigma}{2\pi} X_M^{(\gamma)} (\partial_+ X_M + \gamma^2 \partial_- X_M) = \sum_{k=2}^{\infty} Q_k \gamma^k , \quad (3.10)$$

$$Q_2 = \frac{1}{2} \int_0^{2\pi} \frac{d\sigma}{2\pi} X_{(1)M} \partial_+ X_M = \int_0^{2\pi} \frac{d\sigma}{2\pi} \sqrt{\partial_+ X_N \partial_+ X_N} = \int_0^{2\pi} \frac{d\sigma}{2\pi} \partial_+ t , \quad (3.11)$$

$$Q_k = \frac{1}{2} \int_0^{2\pi} \frac{d\sigma}{2\pi} (X_{(k-1)M} \partial_+ X_M + X_{(k-3)M} \partial_+ X_M) , \quad k \geq 3 . \quad (3.12)$$

Here in (3.11) we used the constraint (3.2). Then  $Q_2$  can be interpreted as the space-time energy: since the general solution for  $t$  is given by (3.3), we conclude that  $Q_2 = \kappa = \mathcal{E}$ . For values of these charges on specific solutions see [20].

One can also define an infinite number of conserved *non-local*  $so(6)$  Lie algebra valued (i.e. *matrix*) currents and associated charges as for any principal or coset sigma model [49, 56]. Let us follow [56] and replace  $X_M$  by an orthogonal  $O(6)$  (or unitary  $SU(4)$ ) matrix

$$g = e^{i\pi P} = 1 - 2P , \quad (3.13)$$

where  $P_{MN} \equiv X_M X_N$  is a projector since  $X_M X_M = 1$ . Then  $g = g^{-1} = 1 - 2P$  and

$$\text{tr} (\partial^a g \partial_a g^{-1}) = 8 \partial^a X_M \partial_a X_M , \quad j_a \equiv g^{-1} \partial_a g = j_a(X) , \quad \partial^a j_a = 0 , \quad (3.14)$$

where the conservation of  $j_a(X)$  given by (3.6) follows from the equations of motion (2.12), (3.1) for  $X_M$ . Defining  $D_a = \partial_a + j_a$  we get  $[\partial_a, D_a] = 0$ . Then starting with the conserved current  $j_a$  one can construct an infinite sequence of conserved non-local currents  $j_a^n$  using the following iterative procedure. Given a conserved current  $j_a^{(n)}$  we define a matrix function  $\chi^{(n)}$  and use it to construct the next conserved current

$$j_a^{(n)} = \epsilon_{ab} \partial^b \chi^{(n)} , \quad j_a^{(n+1)} = D_a \chi^{(n)} , \quad j_a^{(1)} \equiv j_a , \quad \chi^{(0)} = 1 . \quad (3.15)$$

This leads to an infinite set of conserved charges

$$\mathcal{Q}^{(n)} = \int_0^{2\pi} d\sigma j_\tau^{(n)}(\tau, \sigma) . \quad (3.16)$$

For example,

$$\mathcal{Q}_{MN}^{(1)} = \int_0^{2\pi} d\sigma j_{\tau MN}(\tau, \sigma) = 2 \int_0^{2\pi} d\sigma (X_M \partial_\tau X_N - X_N \partial_\tau X_M) , \quad (3.17)$$

is proportional to the  $O(6)$  angular momentum  $J_{MN}$  (2.14), and

$$\partial_\sigma \chi^{(1)} = j_\tau , \quad \chi_{MN}^{(1)}(\tau, \sigma) = \int d\sigma' j_{\tau MN}(\tau, \sigma') , \quad (3.18)$$

$$\mathcal{Q}_{MN}^{(2)} = \int_0^{2\pi} d\sigma' [\partial_\tau + j_\tau(\tau, \sigma')]_{MK} \chi_{KN}^{(1)}(\tau, \sigma') . \quad (3.19)$$

These relations can be consistently defined on an infinite spatial ( $\sigma$ ) line but not on a circle as in the closed string case we are interested in: for  $X_M$  periodic,  $j_a$  (3.6) is also periodic, but its integral in (3.18) may not be, and thus  $\mathcal{Q}^{(2)}$  may not be well-defined (see also related comments in [57, 58]). There are, however, particular classes of solutions (like circular solutions discussed below) for which these charges may be well-defined.



Let us mention also that as in other sigma models with a current satisfying  $j_a = g^{-1}\partial_a g$ ,  $\partial^a j_a = 0$  we can construct a set of chiral currents – symmetric higher spin 2-d local currents which are scalars under  $O(6)$  [59, 60]

$$T_{+...+} = \text{tr } j_+^n, \quad \partial_- T_{+...+} = 0, \quad T_{-...-} = \text{tr } j_-^n, \quad \partial_+ T_{-...-} = 0. \quad (3.20)$$

The special case of spin 2 currents  $T_{++}$  and  $T_{--}$  are the components of the sigma model stress tensor proportional to  $\partial_\pm X_M \partial_\pm X_M$ . There are also other examples of local chiral currents built out of totally symmetric invariant tensors associated with the corresponding Lie algebra [60].

Similar non-local and local currents can be defined [57] also for the full  $AdS_5 \times S^5$  supercoset string sigma model of [3].<sup>†</sup>

To conclude, the above sigma model admits an infinite set of conserved charges which is usually interpreted as implying its integrability [49]. The use of this integrability of a 2-d system for classifying finite energy solutions on a 2-d cylinder is not immediately clear however. In one dimension a system is integrable if it has the same number of commuting integrals of motion as the number of its degrees of freedom. In 2-d where one has infinite number of degrees of freedom it is usually assumed that having infinite set of conserved quantities implies integrability. A more practical definition of integrability could be a prescription of how to construct explicitly a generic solution with required properties. Formal solution-generating techniques (see, e.g., [63]) are not guaranteed a priori to produce finite energy solutions (cf. [64]).

As we shall see below, one can understand the integrability of the  $R_t \times S^5$  sigma model in a very explicit way by reducing it [18, 35] on a special “rotating string” ansatz [14] to a well-known 1-d integrable system – Neumann system [63, 65] or its generalization – Neumann-Rosochatius (NR) system [66, 67].

## 4 Reduction of $R_t \times S^5$ sigma-model to 1-d Neumann system

### 4.1 Rotating string ansatz

Let us consider a string located at the center of spatial part of  $AdS_5$  with time coordinate being proportional to the worldsheet time, i.e. with the  $AdS_5$  coordinates in (2.15) given by

$$Y_5 + iY_0 = e^{it}, \quad Y_1, \dots, Y_4 = 0, \quad t = \kappa\tau. \quad (4.1)$$

The general case when the string can be extended and rotate in both  $AdS_5$  and  $S^5$  will be discussed below in section 6. Since the  $S^5$  metric (2.17) has three commuting

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<sup>†</sup>That the  $AdS_5 \times S^5$  superstring should be integrable being a conformal extension of a bosonic coset sigma model was suggested previously in [47, 61] (see also [62]).

translational isometries in  $\varphi_i$  which give rise to three global commuting integrals of motion  $J_i$  (2.19) to get solutions with non-zero  $J_i$  it is natural to choose the following “rotating string” ansatz for the  $S^5$  coordinates  $X_M$  in (2.15) [14, 18, 35]

$$\begin{aligned} X_1 \equiv X_1 + iX_2 &= z_1(\sigma) e^{iw_1\tau}, & X_2 \equiv X_3 + iX_4 &= z_2(\sigma) e^{iw_2\tau}, \\ X_3 \equiv X_5 + iX_6 &= z_3(\sigma) e^{iw_3\tau}. \end{aligned} \quad (4.2)$$

Here  $w_i$  may be interpreted as frequencies of rotation in the three orthogonal planes. The functions  $z_i$  may be real [18] or, in general, complex [35] and should satisfy  $X_M X_M = 1$ , i.e.

$$z_i = r_i e^{i\alpha_i}, \quad \sum_{i=1}^3 r_i^2 = 1. \quad (4.3)$$

Thus the shape of a rotating string is not changed in time (i.e. the string is rigid) and it always belongs to a 2-sphere.

The closed string periodicity condition (2.10) implies

$$r_i(\sigma + 2\pi) = r_i(\sigma), \quad \alpha_i(\sigma + 2\pi) = \alpha_i + 2\pi m_i, \quad m_i = 0, \pm 1, \pm 2, \dots \quad (4.4)$$

Comparing (4.2) to (2.15) we conclude that the  $S^5$  angles  $\varphi_i$  may depend on both  $\tau$  and  $\sigma$ ,

$$\varphi_i = w_i \tau + \alpha_i(\sigma), \quad (4.5)$$

with the integers  $m_i$  in (4.4) thus playing the role of “winding numbers” in the Cartan directions  $\varphi_i$ .

The space-time energy  $E$  of the string in (2.18) and the spins (2.19) forming a Cartan subalgebra of  $O(6)$  here are given by

$$E = \sqrt{\lambda} \mathcal{E}, \quad \mathcal{E} = \kappa, \quad (4.6)$$

$$J_i \equiv \sqrt{\lambda} \mathcal{J}_i, \quad \mathcal{J}_i = w_i \int_0^{2\pi} \frac{d\sigma}{2\pi} r_i^2(\sigma), \quad \frac{\mathcal{J}_i}{w_i} = 1. \quad (4.7)$$

All other components of the conserved angular momentum tensor  $J_{MN}$  (2.14) vanish automatically if all  $w_i$  are different [18], but their vanishing should be checked if two of the three frequencies are equal.

## 4.2 Integrals of motion and constraints

In general, starting with

$$X_i(\tau, \sigma) = r_i(\tau, \sigma) e^{i\varphi_i(\tau, \sigma)}, \quad (4.8)$$

one finds that the Lagrangian (2.6) reduces to

$$L_S = \frac{1}{2} \sum_{i=1}^3 [\dot{r}_i^2 - r_i'^2 + r_i^2 (\dot{\varphi}_i^2 - \varphi_i'^2)] + \frac{1}{2} \Lambda \left( \sum_{i=1}^3 r_i^2 - 1 \right). \quad (4.9)$$

One can then check that the “rotating string” ansatz (4.2), i.e.

$$r_i = r_i(\sigma) , \quad \varphi_i = w_i \tau + \alpha_i(\sigma) \quad (4.10)$$

is indeed consistent with the equations of motion following from (4.9). Note that because of the formal  $\tau \leftrightarrow \sigma$  symmetry of the 2-d equations of motion another special solution is given by the “pulsating string” ansatz:  $r_i = r_i(\tau)$ ,  $\varphi_i = m_i \sigma + \beta_i(\tau)$ , where  $m_i$  are now integer winding numbers. Then  $r_i(\tau)$  is a solution of a similar Neumann system discussed below (see also section 8).

Substituting (4.2) into (2.6) we get the following effective 1-d “mechanical” system for a particle on a 5-d sphere

$$L = \frac{1}{2} \sum_{i=1}^3 (z'_i z_i'^* - w_i^2 z_i z_i^*) - \frac{1}{2} \Lambda \left( \sum_{i=1}^3 z_i z_i^* - 1 \right) , \quad (4.11)$$

with  $\sigma$  playing the role of time (we changed the sign of  $L$ ). If we set  $z_k = x_k + i x_{k+3}$  ( $k = 1, 2, 3$ ), this is recognized as a special case of the well-known integrable system – the standard  $n = 6$  *Neumann* model [66, 63, 65] describing a harmonic oscillator on a 5-sphere

$$L_N = \frac{1}{2} \sum_{M=1}^6 (x_M'^2 - w_M^2 x_M^2) - \frac{1}{2} \Lambda \left( \sum_{M=1}^6 x_M^2 - 1 \right) . \quad (4.12)$$

Here three of the six frequencies are equal to the other three,  $w_{k+3} = w_k$ ,  $k = 1, 2, 3$ . This implies integrability of the model (4.11) and determines its integrals of motion. Indeed, the Neumann system (4.11) has the following six commuting integrals of motion (see, e.g., [66, 65]):

$$F_M = x_M^2 + \sum_{M \neq N}^6 \frac{(x_M x_N' - x_N x_M')^2}{w_M^2 - w_N^2} , \quad \sum_{M=1}^6 F_M = 1 . \quad (4.13)$$

Since in the present case 3 of the 6 frequencies are equal one needs to consider 3 non-singular combinations of  $F_M$  which then give the 3 integrals of (4.11):

$$I_i = F_i + F_{i+3} , \quad i = 1, 2, 3 , \quad \sum_{i=1}^3 I_i = 1 . \quad (4.14)$$

More explicitly, (4.11) can be written as

$$L = \frac{1}{2} \sum_{i=1}^3 (r_i'^2 + r_i^2 \alpha_i'^2 - w_i^2 r_i^2) - \frac{1}{2} \Lambda \left( \sum_{i=1}^3 r_i^2 - 1 \right) , \quad (4.15)$$

implying that

$$\alpha_i' = \frac{v_i}{r_i^2} , \quad v_i = \text{const} , \quad (4.16)$$

where  $v_i$  are three integrals of motion, which complement the two independent integrals in (4.14). Eliminating  $\alpha_i'$  from (4.15) (changing the sign of the corresponding

term to reproduce the same equations for  $r_i$ ) we then get the following effective Lagrangian for the radial coordinates

$$L = \frac{1}{2} \sum_{i=1}^3 (r_i'^2 - w_i^2 r_i^2 - \frac{v_i^2}{r_i^2}) - \frac{1}{2} \Lambda (\sum_{i=1}^3 r_i^2 - 1) . \quad (4.17)$$

When the new integration constants  $v_i$  vanish, i.e.  $\alpha_i$  are constant, we go back to the case of the  $n = 3$  Neumann model studied in [18]. For non-zero  $v_i$  [35] the Lagrangian (4.17) describes the so called *Neumann-Rosochatius* (NR) integrable system (see, e.g., [67]). As explained above, its integrability follows from the fact that it is a special case of the  $n = 6$  Neumann system, with the integrals of motion (4.14) taking the following explicit form:

$$I_i = r_i^2 + \sum_{j \neq i}^3 \frac{1}{w_i^2 - w_j^2} \left[ (r_i r_j' - r_j r_i')^2 + \frac{v_i^2}{r_i^2} r_j^2 + \frac{v_j^2}{r_j^2} r_i^2 \right] . \quad (4.18)$$

This gives two additional (besides  $v_i$ ) independent integrals of motion.

The conformal gauge constraints (2.7),(2.8) or (3.2) now become

$$\kappa^2 = \sum_{i=1}^3 (r_i'^2 + w_i^2 r_i^2 + \frac{v_i^2}{r_i^2}) , \quad (4.19)$$

$$\sum_{i=1}^3 w_i v_i = 0 . \quad (4.20)$$

As a consequence of (4.20) only two of the three integrals of motion  $v_i$  are independent of  $w_i$ . Note also that (4.19),(4.20) imply  $\kappa^2 = \sum_{i=1}^3 [r_i'^2 + (w_i r_i \pm \frac{v_i}{r_i})^2]$ , so that the space-time energy  $E$  or  $\kappa$  is minimized if  $r_i = \text{const}$  and  $(w_i r_i \pm \frac{v_i}{r_i})^2$  take minimal value, i.e. if  $w_i^2 r_i - \frac{v_i^2}{r_i^3} = 0$ . Since all three  $w_i/v_i$  cannot be positive, this does not mean that  $\kappa$  should vanish. We shall return to the discussion of such solutions below at the end of section 5.1.

To summarize, we are interested in finding periodic finite-energy solitonic solutions of the  $O(6)$  sigma model defined on a 2-cylinder that carry three global charges  $J_i$ . As discussed in [18] (see also below), the periodicity condition (4.4) on  $r_i$  implies that the two integrals of motion  $b_a$  (two appropriate independent combinations of  $I_i$  in (4.18)) can be traded for two integers  $n_a$  labeling different types of solutions. Imposing the periodicity condition (4.4) on  $\alpha_i$  gives, in view of (4.16), the following constraint:

$$v_i \int_0^{2\pi} \frac{d\sigma}{r_i^2(\sigma)} = 2\pi m_i . \quad (4.21)$$

It implies that  $v_i$  should be expressible in terms of the integers  $m_i$ , frequencies  $w_i$  and the “radial” integrals  $b_a$  or  $n_a$  (note also that since the integral in (4.21) is of a positive function,  $m_i = 0$  implies  $v_i = 0$ ). As a result, the moduli space of solutions

will thus be parametrized by  $(w_1, w_2, w_3; n_1, n_2; m_1, m_2, m_3)$ . The constraint (4.20) will give one relation between these 3+2+3 parameters. As a consequence, trading  $w_i$  for the angular momenta using (4.7), the energy of the solutions as determined by (4.6) and the conformal gauge constraint (4.19) will be a function of the  $SO(6)$  spins and the “topological” numbers  $n_a$  and  $m_i$  (cf. (2.24))

$$\mathcal{E} = \mathcal{E}(\mathcal{J}_i; n_a, m_i) , \quad \text{i.e.} \quad E = \sqrt{\lambda} \ \mathcal{E}\left(\frac{J_i}{\sqrt{\lambda}}; n_a, m_i\right) . \quad (4.22)$$

The constraint (4.20) will provide one additional relation between  $J_i$  and  $n_a, m_i$ . Our aim will be to study the relation (4.22) for various types of solutions and in various limits.

### 4.3 Special case of $n = 3$ Neumann system

In the special case of  $v_i = 0$  (when the angles  $\alpha_i$  are constant, i.e.  $\varphi_i$  in (4.10) depend only on  $\tau$ ) the NR system (4.17) reduces to the  $n = 3$  Neumann system with the two independent integrals in (4.18) and only one non-trivial constraint (4.19), which expresses the fact that  $\kappa$  is related to the 1-d Hamiltonian of the Neumann system,

$$H = \frac{1}{2} \sum_{i=1}^3 (r_i'^2 + w_i^2 r_i^2) = \frac{1}{2} \kappa^2 . \quad (4.23)$$

Note that this Hamiltonian is related to the 3 integrals of motion in (4.18) by  $H = \frac{1}{2} \sum_{i=1}^3 w_i^2 I_i$ . Any two of these three integrals are enough to integrate this dynamical system. In order to find the relevant closed string solutions we need also to impose the periodicity condition (4.4), i.e. we are interested in the subsector of periodic solutions of the Neumann model.

Since  $r_i$  belong to a 2-sphere (4.3), the corresponding equations can be expressed in terms of the two  $S^2$  angles, namely,  $\gamma$  and  $\psi$  in (2.15). However, in the general case it is convenient to use another parametrization of  $S^2$  – to replace  $r_i$  by the two “ellipsoidal” coordinates  $\zeta_a$  which are the roots of the equation  $\sum_{i=1}^3 \frac{r_i^2}{\zeta - w_i^2} = 0$ :

$$r_1 = \sqrt{\frac{(\zeta_1 - w_1^2)(\zeta_2 - w_1^2)}{w_{21}^2 w_{31}^2}} , \quad r_2 = \sqrt{\frac{(w_2^2 - \zeta_1)(\zeta_2 - w_2^2)}{w_{21}^2 w_{32}^2}} , \quad (4.24)$$

$$r_3 = \sqrt{\frac{(w_3^2 - \zeta_1)(w_3^2 - \zeta_2)}{w_{31}^2 w_{32}^2}} , \quad w_{ij}^2 \equiv w_i^2 - w_j^2 . \quad (4.25)$$

Expressing the integrals of motion (4.13) in terms of  $\zeta_a$  one finds a system of two 1-st order equations

$$\left(\frac{d\zeta_1}{d\sigma}\right)^2 = -4 \frac{P(\zeta_1)}{(\zeta_2 - \zeta_1)^2} , \quad \left(\frac{d\zeta_2}{d\sigma}\right)^2 = -4 \frac{P(\zeta_2)}{(\zeta_2 - \zeta_1)^2} . \quad (4.26)$$

The function  $P(\zeta)$  is the following 5-th order polynomial

$$P(\zeta) = (\zeta - w_1^2)(\zeta - w_2^2)(\zeta - w_3^2)(\zeta - b_1)(\zeta - b_2) . \quad (4.27)$$

The parameters  $b_1, b_2$  here are the two constants of motion which can be expressed in terms of the integrals  $I_i$  in (4.18) by solving

$$\begin{aligned} b_1 + b_2 &= (w_2^2 + w_3^2)I_1 + (w_1^2 + w_3^2)I_2 + (w_1^2 + w_2^2)I_3 , \\ b_1 b_2 &= w_2^2 w_3^2 I_1 + w_1^2 w_3^2 I_2 + w_1^2 w_2^2 I_3 . \end{aligned} \quad (4.28)$$

The Neumann system Hamiltonian (4.23) is then  $H = \frac{1}{2}(w_1^2 + w_2^2 + w_3^2 - b_1 - b_2) = \frac{1}{2}\kappa^2$ . The polynomial  $P(\zeta)$  in (4.27) can be interpreted as defining a hyperelliptic curve of genus 2 defined by the equation  $s^2 + P(\zeta) = 0$ , with  $s$  and  $\zeta$  being two complex coordinates of  $C^2$ . The formal solution of the system (4.26) is then given in terms of the related  $\theta$ -functions [68, 18].

Thus, the most general three-spin string solutions are naturally associated with special genus 2 hyperelliptic curves [18]. The simpler two-spin case (e.g.,  $w_3 = 0$ ) is associated with an elliptic curve and the corresponding relation between the energy and the spins then involves elliptic functions (see [11, 17, 19]). Elliptic integrals appear also in the one-spin case [37, 8].

The system (4.26) allows one to achieve the full separation of the variables: dividing one equation in (4.26) by the other one can integrate, e.g.,  $\zeta_2$  in terms of  $\zeta_1$  and then obtain a closed equation for  $\zeta_1$  as a function of  $\sigma$ . In finding solutions we need also to take into account the periodicity conditions on  $r_i$  now viewed as conditions on  $\zeta_1, \zeta_2$ . The spins  $\mathcal{J}_i$  in (4.7) expressed in terms of  $\zeta_1, \zeta_2$  satisfy [18]

$$\sum_{i=1}^3 w_i(w_i - \mathcal{J}_i) = \int_0^{2\pi} \frac{d\sigma}{2\pi} (\zeta_1 + \zeta_2) , \quad (4.29)$$

$$\sum_{i=1}^3 \frac{\mathcal{J}_i}{w_i^3} = \frac{1}{w_1^2 w_2^2 w_3^2} \int_0^{2\pi} \frac{d\sigma}{2\pi} \zeta_1 \zeta_2 , \quad \sum_{i=1}^3 \frac{\mathcal{J}_i}{w_i} = 1 . \quad (4.30)$$

To find the energy (4.6) as a function of the spins  $\mathcal{J}_i$  we need to express the frequencies  $w_i$  and the Neumann integrals of motion or  $b_a$  in (4.28) in terms of  $\mathcal{J}_i$ . After finding a periodic solution of (4.26), this reduces to the problem of computing the two independent integrals on the r.h.s. of (4.29) and (4.30).

Let us briefly mention that the case of the NR system (4.17) with  $v_i \neq 0$  can be treated similarly [35]. We can again introduce the ellipsoidal coordinates  $(\zeta_1, \zeta_2)$ , and expressing the integrals of motion (4.18) in terms of  $\zeta_a$  we end up with the same system (4.26) where now

$$\begin{aligned} P(\zeta) &= (\zeta - b_1)(\zeta - b_2)(\zeta - w_1^2)(\zeta - w_2^2)(\zeta - w_3^2) + v_1^2(\zeta - w_2^2)^2(\zeta - w_3^2)^2 \\ &+ v_2^2(\zeta - w_1^2)^2(\zeta - w_3^2)^2 + v_3^2(\zeta - w_1^2)^2(\zeta - w_2^2)^2 . \end{aligned} \quad (4.31)$$

The Hamiltonian of the NR system reduces to  $H = \frac{1}{2}[\sum_{i=1}^3(w_i^2 + v_i^2) - b_1 - b_2]$ . As in the pure Neumann case,  $P(\zeta)$  is the fifth order polynomial which again defines a

hyperelliptic curve  $s^2 + P(\zeta) = 0$ . The general solution of eqs. (4.26) can be again given in terms of theta-functions associated with the Jacobian of the hyperelliptic curve. An example of a solution is provided by the  $v_3 = 0$  case where  $\zeta = w_3^2$  is a root of  $P(\zeta)$  and then the NR system can be solved in terms of the elliptic functions [35].

#### 4.4 Types of solutions and rotating strings in flat space

Let us consider for simplicity the case with  $v_i = 0$  described by the  $n = 3$  Neumann system (general solutions of the NR system have similar structure). The five parameters  $w_i$  (or  $\mathcal{J}_i$ ) and  $b_a$  of the solutions of the Neumann system may be viewed as coordinates on the moduli space of periodic finite-energy solitons. The values of  $b_a$  will not be arbitrary: such solutions will be classified by two integer “winding number” parameters  $n_a$  which will be related to  $w_i$  and  $b_a$  through the periodicity condition. In general, there will be several different solutions for given values of  $\mathcal{J}_i$ , i.e. the energy of the string  $E$  will be a function not only of  $\mathcal{J}_i$  but also of  $n_a$ : there will be a discrete series of solutions with energies starting from some minimal value. Solutions with the same spins  $\mathcal{J}_i$  will be distinguished by values of higher conserved commuting charges.

Depending on the values of these parameters (i.e. location in the moduli space) one may find different geometric types (or shapes) of the resulting rotating string solutions. The shape of the string does not change with time and the string may be “folded” (with topology of an interval) or “circular” (with topology of a circle). A folded string may then be “straight” as in the one- and two-spin examples considered in [8] and [17], or “bent” (at one or several points) as in the general three-spin case [18]. A “circular” string may have the form of a round circle as in the two-spin and three-spin solutions of [14, 15] or may have a more general “bent circle” shape as in the three-spin solutions in [18].

It is useful to review how these different string shapes appear in the case of a closed string rotating in flat  $R^{1,5}$  Minkowski space. In the conformal gauge, string coordinates are then given by solutions of free 2-d wave equation, i.e. by combinations of  $e^{in(\tau \pm \sigma)}$ , subject to the standard constraints (3.2). For a closed string rotating in the two orthogonal spatial 2-planes and moving along the 5-th spatial direction we find (cf. (4.2); here  $t = \kappa\tau$  as in (4.1))

$$X_1 + iX_2 = r_1(\sigma) e^{iw_1\tau}, \quad X_3 + iX_4 = r_2(\sigma) e^{iw_2\tau}, \quad X_5 = p_5\tau, \quad (4.32)$$

$$w_1 = n_1, \quad w_2 = n_2, \quad r_1 = a_1 \sin(n_1\sigma), \quad r_2 = a_2 \sin[n_2(\sigma + \sigma_0)]. \quad (4.33)$$

Here  $\sigma_0$  is an arbitrary integration constant, and  $n_a$  are arbitrary integers. The conformal gauge constraint implies that  $\kappa^2 = p_5^2 + n_1^2 a_1^2 + n_2^2 a_2^2$ . Then the energy, the two spins and the 5-th component of the linear momentum are (here the tension parameter is  $\sqrt{\lambda} \rightarrow \frac{1}{\alpha'}$ )

$$E = \frac{\kappa}{\alpha'}, \quad J_1 = \frac{n_1 a_1^2}{2\alpha'}, \quad J_2 = \frac{n_2 a_2^2}{2\alpha'}, \quad P_5 = \frac{p_5}{\alpha'}, \quad (4.34)$$



i.e.

$$E = \sqrt{P_5^2 + \frac{2}{\alpha'}(n_1 J_1 + n_2 J_2)} . \quad (4.35)$$

To get the two-spin states on the leading Regge trajectory (having minimal energy for given values of the *two* non-zero spins) one is to choose  $n_1 = n_2 = 1$ . The shape of the string depends on the values of  $\sigma_0$  and  $n_1, n_2$ . If  $\frac{\sigma_0}{\pi}$  is irrational then the string always has a “circular” (loop-like) shape. In general, the “circular” string will not be lying in one plane, i.e. will have one or several bends. For rational values of  $\frac{\sigma_0}{\pi}$  the string can be either circular or folded, depending on the values of  $n_1, n_2$ .

Let  $\sigma_0 = 0$ . If  $n_1 = n_2$  the string is folded and straight, i.e. have no bends. Indeed, then  $X_1 + iX_2$  is proportional to  $X_3 + iX_4$  and thus one may put the string in a single 2-plane by a global  $O(4)$  rotation. If both  $n_1$  and  $n_2$  are either even or odd and different then the string is folded and has several bends (in the 13 and 24 planes). For example, if  $n_2 = 3n_1$  then the folded string is wound  $n_1$  times and has two bends (for  $a_1 = a_2$  we have  $r_2 = r_1(3 - 4r_1^2)$ ). Next, let us choose  $\sigma_0 = \frac{\pi}{2n_2}$ . Then for  $n_1 = n_2$  the string is an ellipsoid, becoming a round circle in the special case of  $a_1 = a_2$  (i.e.  $\mathcal{J}_1 = \mathcal{J}_2$ ) [14]. The string is also circular if  $n_1$  is even and  $n_2$  is odd. If, however,  $n_1$  is odd and  $n_2$  is even the string is folded, e.g., if  $n_2 = 2n_1$  then the folded string is wound  $n_1$  times and has a single bend at one point.

The structure of spinning string soliton solutions in curved  $S^5$  case is analogous. The equations of motion of the Neumann system are linearized on the Jacobian of the hyperelliptic curve. The image of the string in the Jacobian whose real connected part is identified with the Liouville torus can wind around two non-trivial cycles with the winding numbers  $n_1$  and  $n_2$  respectively [18]. The size and the shape of the Liouville torus are governed by the moduli  $(w_i, b_a)$ . Specifying the winding numbers  $n_1, n_2$ , two of the five parameters  $(w_i, b_a)$  are then uniquely determined by the periodicity conditions. The actual rigid shape of the physical string lying on two-sphere will depend on the numbers  $n_1, n_2$  and on the remaining moduli parameters (relative values of  $b_a$  and  $w_i^2$ ): it may be of (bent) folded type or of (deformed) circular type. Various examples of folded and circular 3-spin string solutions and their energies were discussed in [14, 18]. In most 3-spin cases finding explicit relation for the energy (4.22) is complicated, but one can always develop large  $\mathcal{J}_i$  perturbation theory [18]. We shall discuss some examples of such solutions below.

Finally, let us note that while the Neumann or NR 1-d systems have a small finite number of commuting integrals, there are infinite commuting conserved charges in the original 2-d sigma model and the corresponding integrable spin chain on the SYM side. These are expressed in terms of the NR integrals in the present case, see [20] for details.

## 5 Simplest circular solutions in $R_t \times S^5$ : $\Lambda = \text{const}$

A simple special class of solutions of the system (4.11) or (4.17) is found by demanding that the Lagrange multiplier  $\Lambda$  in (2.12) is constant, i.e.  $\dot{X}_M^2 - X_M'^2 = \text{const}$ . In this case the radii  $r_i$  turn out to be constant (and  $n_a = 0$ , i.e. there are no bends). This represents an interesting new class of circular 3-spin solutions [35] which includes as a special case the circular solution of [14] where two out of three spins were equal.

### 5.1 Constant radii solution

Let us start with the Lagrangian (4.11) written in terms of 3 complex coordinates  $z_i$ . Then the equations of motion are

$$z_i'' + m_i^2 z_i = 0, \quad m_i^2 \equiv w_i^2 + \Lambda, \quad \sum_{i=1}^3 |z_i|^2 = 1, \quad (5.1)$$

$$\Lambda = \sum_{i=1}^3 (|z_i'|^2 - w_i^2 |z_i|^2). \quad (5.2)$$

Eq. (5.1) can be easily integrated if one assumes that  $\Lambda = \text{const}$ ,

$$z_i = a_i e^{im_i \sigma} + b_i e^{-im_i \sigma}, \quad (5.3)$$

where  $a_i, b_i$  are complex coefficients. The periodicity condition  $z_i(\sigma + 2\pi) = z_i(\sigma)$  implies that  $m_i$  must be integer. It is easy to show [35] that modulo the global  $SU(3) \in SO(6)$  invariance the solution of (5.3) that satisfies both  $\Lambda = \text{const}$  and  $\sum_{i=1}^3 |z_i|^2 = 1$  should have  $b_i = 0$  (or  $a_i = 0$ ), i.e. should look like ( $m_i$  may be positive or negative and  $a_i$  may be made real by  $U(1)$  rotations)

$$z_i = a_i e^{im_i \sigma}, \quad \sum_{i=1}^3 a_i^2 = 1. \quad (5.4)$$

It may seem that one may get a new solution if two of the windings  $m_i$  are equal while the third is zero, i.e.  $z_1 = a \cos m\sigma$ ,  $z_2 = a \sin m\sigma$ ,  $z_3 = \sqrt{1 - a^2}$  (which is, in fact, the circular solution of [14]), but this configuration can be transformed into the form (5.4) by a global  $SU(2)$  rotation.

One can also rederive (5.4) by starting with (4.17), (4.16). The potential  $w_i r_i^2 + \frac{v_i^2}{r_i^2}$  in (4.17) has a minimum, and that suggests that  $r_i = \text{const}$  may be a solution. The equations of motion that follow from (4.17)

$$r_i'' = -w_i^2 r_i + \frac{v_i^2}{r_i^3} - \Lambda r_i, \quad (5.5)$$

$$\Lambda = \sum_{j=1}^3 \left( r_j'^2 - w_j^2 r_j^2 + \frac{v_j^2}{r_j^2} \right), \quad \sum_{j=1}^3 r_j^2 = 1 \quad (5.6)$$

are indeed solved by

$$r_i(\sigma) = a_i = \text{const} , \quad w_i^2 - \frac{v_i^2}{a_i^4} = \nu^2 = \text{const} , \quad \Lambda = -\nu^2 , \quad (5.7)$$

where  $\nu$  is an arbitrary constant (which may be positive or negative). Eq. (5.7) then implies  $a_i^2 = \frac{|v_i|}{\sqrt{w_i^2 - \nu^2}}$ ,  $\alpha'_i = \frac{v_i}{a_i^2} = \frac{v_i}{|v_i|} \sqrt{w_i^2 - \nu^2} \equiv m_i$ , i.e.  $\alpha_i = \alpha_{0i} + m_i \sigma$ , where  $m_i$  must be integer to satisfy the periodicity condition (4.4) and  $\alpha_{0i}$  may be set to zero by independent  $SO(2)$  rotations. Then

$$w_i^2 = m_i^2 + \nu^2 , \quad v_i = a_i^2 m_i , \quad \sum_{i=1}^3 a_i^2 = 1 . \quad (5.8)$$

The constraints (4.19),(4.20) give  $\kappa^2 = 2 \sum_{i=1}^3 a_i^2 w_i^2 - \nu^2$ , and  $\sum_{i=1}^3 a_i^2 w_i m_i = 0$ . As a result, we get the following relations for the energy and spins [35] (cf. (4.6),(4.7))

$$\mathcal{E}^2 = 2 \sum_{i=1}^3 \sqrt{m_i^2 + \nu^2} \mathcal{J}_i - \nu^2 , \quad (5.9)$$

$$\sum_{i=1}^3 \frac{\mathcal{J}_i}{\sqrt{m_i^2 + \nu^2}} = 1 , \quad (5.10)$$

$$\sum_{i=1}^3 m_i \mathcal{J}_i = 0 . \quad (5.11)$$

We shall assume for definiteness that all  $w_i$  and thus all  $\mathcal{J}_i$  are non-negative. Then (5.11) implies that one of the three  $m_i$ 's must have the opposite sign to the other two. One can check directly that the only non-vanishing components of the  $SO(6)$  angular momentum tensor  $J_{MN}$  (2.14) on this solution are indeed the Cartan ones  $J_i$  in (2.19).

The special case of  $\nu^2 = 0$  (or  $\Lambda = 0$ ) corresponds to a solution for the string in flat space which can be embedded into  $S^5$  by choosing the free radial parameters of a circular string to satisfy the condition  $\sum_{i=1}^3 a_i^2 = 1$ . Indeed, as follows from (5.8) for  $\nu^2 = 0$  we find that all frequencies must be integer  $w_i = |m_i|$ . We may choose, e.g.,  $m_1 < 0$ ,  $m_2 > 0$ ,  $m_3 > 0$ , so that the solution is a combination of the left and right moving waves in different directions (we use complex combinations of the coordinates in (4.2), cf. (4.32),(4.33))

$$X_1 = a_1 e^{im_1(\sigma - \tau)} , \quad X_2 = a_2 e^{im_2(\sigma + \tau)} , \quad X_3 = a_3 e^{im_3(\sigma + \tau)} , \quad \sum_{i=1}^3 a_i^2 = 1 . \quad (5.12)$$

Here we get from (5.9)–(5.11)

$$\mathcal{E}^2 = 2 \sum_{i=1}^3 |m_i| \mathcal{J}_i , \quad \sum_{i=1}^3 \frac{\mathcal{J}_i}{|m_i|} = 1 , \quad \sum_{i=1}^3 m_i \mathcal{J}_i = 0 . \quad (5.13)$$

This corresponds to a very special point in the moduli space of solutions. For fixed  $m_i$ , we get two constraints on  $\mathcal{J}_i$ , and the energy is given by the standard flat-space Regge relation (cf. (4.35)). Then  $|m_1|\mathcal{J}_1 = m_2\mathcal{J}_2 + m_3\mathcal{J}_3$  (where  $\mathcal{J}_2$  and  $\mathcal{J}_3$  are related via  $\sum_{i=1}^3 \frac{\mathcal{J}_i}{|m_i|} = 1$ ) and thus  $\mathcal{E}^2 = 4|m_1|\mathcal{J}_1$ . The energy of this “flat” solution thus does *not* have a regular (1.2) expansion in integer powers of  $\frac{1}{\mathcal{J}^2} = \frac{\lambda}{J^2}$ ,  $\mathcal{J} = \sum_{i=1}^3 \mathcal{J}_i$ . This will no longer be so in the genuinely “curved”  $\nu \neq 0$  case where we will have indeed a regular expansion for the energy in  $\frac{1}{\mathcal{J}^2}$ , as in the case of the circular solution of [14]. This then opens up a possibility of direct comparison with perturbative anomalous dimensions in SYM theory.

## 5.2 Energy as function of spins

In general, to express  $\mathcal{E}$  in terms of  $\mathcal{J}_i$  and  $m_i$  one is first to solve the condition (5.10) in terms of  $\nu$ , determining  $\nu$  as a function of  $\mathcal{J}_i$  and  $m_i$  and then substitute the result into (5.9). The condition (5.11) may be imposed at the very end, implying that for given spins  $\mathcal{J}_i$  the solution exists only for a special choice of the integers  $m_i$ . Expanding in large total spin  $\mathcal{J} = \sum_{i=1}^3 \mathcal{J}_i$  as in [14, 18] one finds [35] that  $\nu^2 = \mathcal{J}^2 - \sum_{i=1}^3 m_i^2 \frac{\mathcal{J}_i}{\mathcal{J}} + \dots$  and thus

$$\mathcal{E} = \mathcal{J} + \frac{1}{2\mathcal{J}} \sum_{i=1}^3 m_i^2 \frac{\mathcal{J}_i}{\mathcal{J}} + \dots \quad (5.14)$$

Thus, as in other examples in [14, 17, 18], here the energy admits a regular expansion in  $\frac{1}{\mathcal{J}^2} = \frac{\lambda}{J^2}$  as in (1.2)

$$E = J(1 + \frac{\lambda}{2J^2} \sum_{i=1}^3 m_i^2 \frac{J_i}{J} + \dots) = J + \frac{\lambda}{2J} \sum_{i=1}^3 m_i^2 \frac{J_i}{J} + \dots \quad (5.15)$$

where  $m_i$  should satisfy the constraint  $\sum_{i=1}^3 m_i J_i = 0$ .

Let us now look at some special cases. In the one-spin case  $(0, 0, J_3)$ , i.e.  $\mathcal{J}_1 = \mathcal{J}_2 = 0$ ,  $a_1 = a_2 = 0$ , we have  $w_3^2 = \nu^2$ , i.e.  $m_3 = 0$  and  $J_3 = w_3$ , and then  $\mathcal{E} = \mathcal{J}_3$ . This is simply the point-like BMN geodesic case: there is no  $\sigma$ -dependence.

In the two-spin case  $(J_1, J_1, 0)$ , i.e.  $\mathcal{J}_3 = 0$ ,  $a_3 = 0$ , the equation (5.10) for  $\nu^2$  becomes a quartic equation. Its simple explicit solution is found in the equal-spin case when  $\mathcal{J}_1 = \mathcal{J}_2$ , i.e. when

$$a_1 = a_2 = \frac{1}{\sqrt{2}} \quad , \quad m_2 = -m_1 \equiv m > 0 \quad , \quad (5.16)$$

so that

$$\mathcal{E} = \sqrt{\mathcal{J}^2 + m^2} \quad , \quad \mathcal{J} \equiv \mathcal{J}_1 + \mathcal{J}_2 = 2\mathcal{J}_2 \quad , \quad (5.17)$$

i.e.

$$E = J \sqrt{1 + m^2 \frac{\lambda}{J^2}} \quad . \quad (5.18)$$

We get

$$X_1 = \frac{1}{\sqrt{2}} e^{i w \tau - i m \sigma}, \quad X_2 = \frac{1}{\sqrt{2}} e^{i w \tau + i m \sigma}, \quad w = \sqrt{\nu^2 + m^2}. \quad (5.19)$$

This solution is thus equivalent to the circular 2-spin solution of [14] – it is related to it by an  $SO(4)$  rotation:  $X'_1 = \frac{1}{\sqrt{2}}(X_1 + X_2)$ ,  $X'_2 = \frac{1}{\sqrt{2}}(-X_1 + X_2)$ . In the general case of two unequal spins we can again solve (5.10) in the limit of large  $\mathcal{J}_1, \mathcal{J}_2$  (for fixed  $m_1, m_2$ ), getting the special case of (5.14) with  $m_1 \mathcal{J}_1 + m_2 \mathcal{J}_2 = 0$ ,  $\mathcal{J}_3 = 0$ .

Another special case is  $(J_2, J_2, J_3)$  when two out of three non-vanishing spins are equal, e.g.,  $\mathcal{J}_1 = \mathcal{J}_2$ . Setting

$$m_3 = 0, \quad m_1 = -m_2 = m, \quad a_3 = a < 1, \quad a_1 = a_2 = \sqrt{1 - a^2}, \quad (5.20)$$

$$\mathcal{J}_3 = a^2 \nu, \quad \mathcal{J}_1 = \mathcal{J}_2 = \frac{1}{2}(1 - a^2)\sqrt{m^2 + \nu^2}, \quad (5.21)$$

we thus find from (5.14)

$$\mathcal{E} = \mathcal{J} + \frac{m^2 \mathcal{J}_2}{\mathcal{J}^2} + \dots, \quad \text{i.e.} \quad E = J + \frac{m^2 \lambda J_2}{J^2} + \dots, \quad J = 2J_2 + J_3. \quad (5.22)$$

This solution is equivalent to the circular 3-spin solution with two equal spins in [14, 15] (the two backgrounds are related by a global rotation in  $X_2, X_3$  directions converting  $e^{im\sigma}$  into  $\cos m\sigma$  and  $\sin m\sigma$ ). The corresponding operator in the gauge theory  $\text{tr}(X^{J_1} Y^{J_1} Z^{J_3}) + \dots$  (belonging to the  $SO(6)$  representation with Dynkin indices  $[J_2 + J_3, 0, J_2 - J_3]$  for  $J_2 > J_3$ ) which has the 1-loop anomalous dimension equal to (5.22) does indeed exist as was found in [21].

More generally, we may consider a 3-spin solution  $(J_2, J_2, J_3)$  with  $m_3 \neq 0$ , so that  $(m_1 + m_2)J_2 + m_3 J_3 = 0$ . Then (5.15) gives

$$\mathcal{E} = \mathcal{J} + \frac{\mathcal{J}_2}{2\mathcal{J}^2} [m_1^2 + m_2^2 + (m_1 + m_2)^2 \frac{\mathcal{J}_2}{\mathcal{J}_3}] + \dots, \quad (5.23)$$

which generalizes (5.22) to the case when  $m_1 + m_2 \neq 0$ . The energy is minimal in the latter case. This suggests that the band of such states in the same representation  $[J_2 + J_3, 0, J_2 - J_3]$  (if  $|\frac{m_3}{m_1 + m_2}| > 1$ ) but with higher energy than (5.22) should also be found on the SYM side.

To summarize, the constant-radii solutions of the NR system represent a simple generalization of the circular 2-spin and 3-spin solutions of [14] which have regular expansion of the energy in powers of  $\frac{\lambda}{J^2}$ . Therefore, it should be possible to match, as in [16, 18, 19, 21], the coefficient of the  $O(\lambda)$  term in (5.14) with the SYM anomalous dimensions determined by the Hamiltonian of the integrable  $SU(2, 2|4)$  spin chain [22, 23] in the corresponding 3-spin subsector of states.

### 5.3 Quadratic fluctuations near circular solutions

The remarkable simplicity of the circular solutions discussed above makes it easy to find the quadratic fluctuation action and to compute the corresponding spectrum of string fluctuations. This in turn allows one to analyse the stability of the solution and to find the string 1-loop correction to the ground-state energy, in the same way as this was done in [15] for a particular 3-spin circular solution with two equal spins (5.21). In spite of the  $\sigma$ -dependence of the solution, the quadratic action turns out to have constant coefficients, just like in the BMN case [4, 6] when one expands near the point-like geodesic in  $S^5$  [8, 11]. Sending  $J \rightarrow \infty$  for fixed  $\frac{\lambda}{J^2} \ll 1$  suppresses higher loop corrections to masses of excited string states. As a result, as in the “plane-wave” BMN case, the string fluctuation spectrum can be found *exactly*.

To illustrate this, here we shall consider the bosonic part of the quadratic fluctuation action following [35]. The fermionic part of the spectrum can be easily found in the same way as was done (in a special case (5.21)) in a [15]. In contrast to the BMN case, here we are expanding near a non-supersymmetric solution, and the resulting world-sheet string action (in the static or light-cone type gauge) will not have a world-sheet supersymmetry. There remains an interesting question if a “nearly-BPS” property of similar rotating string solutions in the  $\lambda \rightarrow 0$  limit observed in [42] imposes certain constraints on the world-sheet action.

It is straightforward to find the quadratic fluctuation Lagrangian by expanding near the solution (5.4) or (5.7)–(5.8) following [15, 35]. Using 3 complex combinations of coordinates in (4.2) and expanding ( $X_i \rightarrow X_i + \tilde{X}_i$ ) the sigma model action (2.5) near the classical solution (5.4), i.e.

$$X_i = a_i e^{i w_i \tau + i m_i \sigma} , \quad w_i^2 = \sqrt{m_i^2 + \nu^2} , \quad \sum_{i=1}^3 a_i^2 = 1 , \quad \sum_{i=1}^3 a_i^2 w_i m_i = 0 \quad (5.24)$$

we find the following Lagrangian for the quadratic fluctuations (see [15])

$$\tilde{L} = -\frac{1}{2} \partial_a \tilde{X}_i \partial^a \tilde{X}_i^* + \frac{1}{2} \Lambda \tilde{X}_i \tilde{X}_i^* , \quad (5.25)$$

where  $\Lambda = -\nu^2$  (see (5.7)) and  $\tilde{X}_i$  are subject to the constraint\*

$$\sum_{i=1}^3 (X_i \tilde{X}_i^* + X_i^* \tilde{X}_i) = 0 . \quad (5.26)$$

To solve this constraint we set

$$\tilde{X}_i = e^{i w_i \tau + i m_i \sigma} (g_i + i f_i) , \quad (5.27)$$

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\*The imposition of the conformal gauge constraints on the fluctuations is not necessary in order to determine the non-trivial part of the fluctuation spectrum [14, 15] (solving the constraints in terms of fluctuation of  $t$  leads to equivalent result [15]). In addition to  $S^5$  fluctuations there are also  $AdS_5$  fluctuations: one massless and 4 massive ones with mass  $\kappa$  coming from the classical value of the Lagrange multiplier  $\tilde{\Lambda}$  [14, 15].

where  $g_i$  and  $f_i$  are real functions of  $\tau$  and  $\sigma$ . Then (5.26) reduces to

$$\sum_{i=1}^3 a_i g_i = 0 . \quad (5.28)$$

Using (5.27) the Lagrangian (5.25) becomes (after integrating by parts, cf. [15])

$$\tilde{L} = \sum_{i=1}^3 \left[ \frac{1}{2} (\dot{f}_i^2 + \dot{g}_i^2 - f_i'^2 - g_i'^2) - 2w_i f_i \dot{g}_i + 2m_i f_i g_i' \right] . \quad (5.29)$$

To solve the linear relation (5.28) we may apply a global  $O(3)$  rotation to  $g_i$ ,  $\bar{g}_i = M_{ij}(a)g_j$ , which preserves the kinetic terms in (5.29) and transforms  $\sum_{i=1}^3 a_i g_i$  into  $\bar{g}_1$ ; then we may set the latter to zero in the resulting Lagrangian (5.29). Equivalently, we may solve (5.28) directly for  $g_1$  and substitute it into (5.29). The result (after diagonalization) is a special case of the following 2-d Lagrangian (summation over  $p, q$  is assumed)

$$L = \frac{1}{2} \dot{x}_p^2 - \frac{1}{2} x_p'^2 + F_{pq} x_p \dot{x}_q - H_{pq} x_p x_q' , \quad (5.30)$$

where  $x_p = (f_1, f_2, f_3, g_2, g_3)$  and  $F_{pq}$  and  $H_{pq}$  are constant antisymmetric matrices depending on  $a_i, w_i, m_i$ . Eq. (5.30) can be written also as (ignoring total derivative)

$$L = \frac{1}{2} (\dot{x}_p + F_{pq} x_q)^2 - \frac{1}{2} (x_p' + H_{pq} x_q)^2 - (F_{pq} F_{qk} - H_{pq} H_{qk}) x_p x_k , \quad (5.31)$$

i.e. it represents a massive scalar 2-d theory coupled to a constant 2-d gauge field (which can be “rotated away” at the expense of making the mass term  $\tau$  and  $\sigma$  dependent). The Lagrangian (5.30) can be also interpreted as a *light-cone gauge* ( $u = \tau$ ) Lagrangian for the bosonic string sigma model  $L = -(\eta^{ab} g_{mn} + \epsilon^{ab} B_{mn}) \partial_a x^m \partial_b x^n$  in (in general, non-conformal) *plane-wave* background with the following metric and antisymmetric 2-form field

$$ds^2 = 2dudv + 2F_{pq} x_p dx_q du + dx_p dx_p , \quad B_2 = 2H_{pq} x_p dx_q \wedge du . \quad (5.32)$$

By analogy with the BMN case, one may say that the geometry “seen” in the large  $\mathcal{J}$  limit by the circular rotating string is a generalized plane-wave background. The resulting quadratic string excitation spectrum for such an action can be found in a more or less explicit way (as in [70]).

For example, let us consider the 2-spin case where  $m_3 = 0$  and

$$a_1^2 + a_2^2 = 1 , \quad a_3 = 0 , \quad a_1^2 m_1 w_1 + a_2^2 m_2 w_2 = 0 , \quad w_1^2 - m_1^2 = w_2^2 - m_2^2 = \nu^2 . \quad (5.33)$$

We shall assume that  $w_i > 0$ ,  $m_1 < 0$ ,  $m_2 > 0$ . In this case  $f_3, g_3$  decouple (they have mass  $\nu$ , cf. (5.25)) and we get the following Lagrangian for the remaining three  $x_s$ -fluctuations  $f_1, f_2$  and (rescaled)  $g_2$

$$\begin{aligned} \tilde{L} = & \frac{1}{2} (\dot{f}_1^2 + \dot{f}_2^2 + \dot{g}_2^2 - f_1'^2 - f_2'^2 - g_2'^2) \\ & + 2(a_2 w_1 f_1 - a_1 w_2 f_2) \dot{g}_2 - 2(a_2 m_1 f_1 - a_1 m_2 f_2) g_2' . \end{aligned} \quad (5.34)$$



To find the spectrum of characteristic frequencies corresponding to this action we note that since  $f_i$  and  $g_i$  must be periodic in  $\sigma$  one can expand the solution of the quadratic fluctuation equations in modes

$$x_s = \sum_{n=-\infty}^{\infty} \sum_{k=1}^8 A_{sn}^{(k)} e^{i(\Omega_{n,k}\tau + n\sigma)} , \quad (5.35)$$

where  $k$  labels different frequencies for a given value of  $n$  (we shall suppress the index  $k$  below). Plugging this into the classical equations that follow from (5.34) one finds the following equation for the 4 non-trivial characteristic frequencies (it expresses the vanishing of determinant of the characteristic matrix)

$$(\Omega^2 - n^2)^2 - 4a_2^2(w_1\Omega - m_1n)^2 - 4a_1^2(w_2\Omega - m_2n)^2 = 0 . \quad (5.36)$$

The stability condition is that all 4 roots should be real. The solutions are obviously real for  $n = 0$  so an instability may appear only for  $n = \pm 1, \dots$ . In the special case of the equal-spin circular solution of [14], i.e. (5.16), (5.17), we find  $(\Omega^2 - n^2)^2 - 4w^2\Omega^2 - 4m^2n^2 = 0$ , i.e. [14]

$$\Omega_{\pm}^2 = n^2 + 2\nu^2 + 2m^2 \pm 2\sqrt{(\nu^2 + m^2)^2 + n^2(\nu^2 + 2m^2)} , \quad (5.37)$$

which implies instability when  $n^2 - 4m^2 < 0$ , i.e. for  $n = \pm 1, \dots, \pm(2m-1)$  [15]. This instability is present also for generic 2-spin solutions with  $a_1 \neq a_2$ ,  $m_1 \neq -m_2$ .

In spite of the instability it is useful to work out the spectrum of frequencies in the limit of large spins (i.e. large  $\nu$ , cf. (5.21)) since the resulting energies may be compared to SYM theory. The large  $\nu$  expansion of (5.37) gives (for the lower-energy modes)

$$\Omega_{\pm} = \pm \frac{1}{2\nu} n \sqrt{n^2 - 4m^2} + O\left(\frac{1}{\nu^3}\right) , \quad (5.38)$$

and so the contribution to the energy of rotating string from (a pair of) such modes is (here  $\kappa^2 = \nu^2 + 2m^2$ ,  $J = J_1 + J_2 = \sqrt{\lambda} \sqrt{\nu^2 + m^2}$ )

$$\Delta E_n = \frac{2|\Omega_{-}|}{\kappa} = \frac{1}{\nu^2} n \sqrt{n^2 - 4m^2} + O\left(\frac{1}{\nu^4}\right) = \frac{\lambda}{J^2} n \sqrt{n^2 - 4m^2} + O\left(\frac{\lambda^2}{J^4}\right) . \quad (5.39)$$

This expression was indeed reproduced [16] on the SYM side (for  $m = 1$ ) as the anomalous dimension of excited string states corresponding to a particular Bethe root distribution of a Heisenberg spin chain related to the dilatation operator in the two R-charge sector.

In the general  $(m_1, m_2)$  case, there are modes that have  $\Omega \sim \frac{1}{\nu}$  and modes for which  $\Omega^2 \rightarrow 4\nu^2$  at large  $\nu$  (see [15]). Expanding (5.36) at large  $\nu$  assuming  $\Omega = O(\frac{1}{\nu})$  we find the following generalization of (5.38)

$$\Omega_{\pm} = \frac{1}{2\nu} n \left[ 2a_2^2 m_1 + 2a_1^2 m_2 \pm \sqrt{n^2 - 4a_1^2 a_2^2 (m_1 - m_2)^2} \right] + O\left(\frac{1}{\nu^3}\right) , \quad (5.40)$$

where  $a_1^2 + a_2^2 = 1$ . Eq. (5.40) reduces to (5.38) in the equal-spin case when  $a_1^2 = a_2^2 = \frac{1}{2}$ ,  $m_1 = -m_2$ . Recalling that we have the constraint  $m_1 J_1 + m_2 J_2 = 0$  where  $J_i = a_i^2 \sqrt{m_i^2 + \nu^2}$ , one concludes about the existence of unstable modes with  $n^2 < 4|m_1 m_2|$  [35]. Again, one should be able to reproduce the analog of (5.39) in the case of (5.40) on the gauge theory side.

It is straightforward to find the generalisation of (5.34),(5.36) to the 3-spin case, i.e. when  $a_3$  is non-zero. The resulting spectrum is similar to the spectrum in the  $(J_1, J_2 = J_3)$  case in [15]. The generalization of the eq.(5.36) to the 3-spin case is [35]

$$\begin{aligned} (\Omega^2 - n^2)^4 &- (\Omega^2 - n^2)^2[(a_2^2 + a_3^2)\tilde{\Omega}_1^2 + (a_2^2 + a_3^2)\tilde{\Omega}_2^2 + (a_1^2 + a_2^2)\tilde{\Omega}_3^2] \\ &+ a_3^2\tilde{\Omega}_1^2\tilde{\Omega}_2^2 + a_2^2\tilde{\Omega}_1^2\tilde{\Omega}_3^2 + a_1^2\tilde{\Omega}_2^2\tilde{\Omega}_3^2 = 0 , \end{aligned} \quad (5.41)$$

where  $\tilde{\Omega}_i \equiv 2(w_i \Omega - m_i n)$  and  $a_i$  and  $w_i$  can be expressed in terms of  $\nu$  and  $m_i$  using (5.24). Setting  $\tilde{\Omega}_3 = 0$ ,  $a_3 = 0$  leads us back to (5.36). Eq. (5.41) gives 8 characteristic frequencies, 4 of which scale as  $\frac{1}{\nu}\bar{\Omega}$  in the large  $\nu$  (large  $\mathcal{J}$ ) limit. In general, there is a range of parameters for which the solution is stable [15, 35], i.e.  $\Omega$ 's are real.

For example, for the choice of the parameters in (5.20) when two of the spins are equal, we find [15] ( $\Omega \rightarrow \frac{1}{\nu}\bar{\Omega}$ )

$$\bar{\Omega}^2 = \frac{1}{4}n^2 \left[ n^2 + 2(3a^2 - 1)m^2 \pm 2m\sqrt{(3a^2 - 1)^2 m^2 + 4a^2(n^2 - m^2)} \right]. \quad (5.42)$$

Note that the limit  $m = 0$  corresponds to the point-particle (BMN) case when  $\Omega = \sqrt{\nu^2 + n^2}$ . The condition of stability, i.e.  $\Omega^2 \geq 0$  is obtained by demanding that  $(p^2 - 4)(p^2 - 4a^2) \geq 0$  and  $(3a^2 - 1)^2 + 4a^2(p^2 - 1) \geq 0$ , where  $p \equiv \frac{n}{m}$ . For  $m = 1$  the stability condition is satisfied if  $a^2 \geq \frac{1}{4}$  [15]. Similar stability conditions on  $a$  (or  $\cos \gamma_0$  in the notation of [15]) are found for other values of  $m$  [15, 35].

## 5.4 1-loop string correction to the classical energy

As was shown in [15], for the stable 3-spin solution (5.20) one can compute the 1-loop correction to the classical energy (5.22) by summing over all (bosonic and fermionic) fluctuation frequencies. As in the static gauge, here  $t = \kappa\tau$  and so the space-time energy and the 2-d energy (sum of  $\frac{1}{2}\omega_n$  for all oscillator frequencies) are related by [11, 15]  $E = \frac{1}{\kappa}E_{2-d}$ . Thus the 1-loop correction is given by the standard sum of the oscillator frequencies

$$E = \frac{1}{\kappa}E_{2-d} = \frac{1}{2\kappa} \left( \sum_{n \in \mathbb{Z}} \omega_n^B - \sum_{r \in \mathbb{Z} + \frac{1}{2}} \omega_r^F \right) , \quad (5.43)$$

where  $\omega_n^B = \sum_{k=1}^8 \Omega_{n,k}^B$  and  $\omega_r^F = \sum_{k=1}^8 \Omega_{r,k}^F$  and the index  $k$  labels the characteristic frequencies. Here we need also to include contributions of  $AdS_5$  fluctuations with

masses equal to  $\kappa$  [15]. As expected on the basis of conformal invariance of the  $AdS_5 \times S^5$  string theory, this expression is found to be UV finite [15]. The 1-loop correction vanishes in the “point-particle” limit when  $m = 0$  in (5.20), in agreement with the non-renormalization of the energy of the corresponding BPS state dual to a gauge theory operator with protected conformal dimension [4].

As was found in [15], the leading term in  $E_1$  in the large  $\kappa \approx \nu \approx \mathcal{J}$  limit is given for  $m = 1$  by

$$E_1 = \frac{1}{\kappa^2} d_1 + O\left(\frac{1}{\kappa^3}\right), \quad (5.44)$$

$$d_1 = -\frac{1}{2}[5a^2 + 4 - \sqrt{3(4a^2 - 1)} - 4\sqrt{3a^2 + 1}]. \quad (5.45)$$

We are interested in the limit when  $J_i \rightarrow \infty$  with  $\frac{J_i}{J}$  held fixed (here  $J = \sum_{i=1}^3 J_i = J_1 + 2J_2$ ). Since at large  $\kappa$  we have  $\frac{1}{\kappa^2} = \frac{\lambda}{J^2} + \dots$ , and (see (5.21))  $a = a(\frac{J_2}{J}) \approx 1 - \frac{J_2}{J} \geq \frac{1}{2}$  we get (cf. (1.5))

$$E_1 = \frac{\lambda}{J^2} d_1\left(\frac{J_2}{J}\right) + \dots \quad (5.46)$$

For  $J \gg J_2$  we find  $d_1(\frac{J_2}{J}) \approx 1 - \frac{7J_2}{J}$ . Combining this with the classical result for the energy (5.22), we obtain [15]

$$E = J + \frac{\lambda}{J^2} [J_2 + d_1\left(\frac{J_2}{J}\right) + \dots] + \dots = J + \frac{\lambda}{J} \left[ \frac{J_2}{J} + \frac{1}{J} d_1\left(\frac{J_2}{J}\right) + \dots \right] + \dots, \quad (5.47)$$

i.e.

$$E = J + \frac{\lambda}{J} \left[ \frac{J_2}{J} + \frac{1}{J} \left[ 1 - \frac{7J_2}{J} + O\left(\left(\frac{J_2}{J}\right)^2\right) \right] + O\left(\frac{1}{J^2}\right) \right] + O\left(\frac{\lambda^2}{J^3}\right). \quad (5.48)$$

We conclude that the leading order  $J$  term in the classical energy is not modified by the 1-loop correction, and that the 1-loop contribution to the first classical correction term  $\frac{\lambda}{J}$  is subleading in the  $\frac{1}{J}$  expansion, in agreement with (1.6).

It is natural to conjecture that all higher-loop sigma model superstring corrections are also subleading at large  $J$ . As in the BMN case (see [11] and sect. 3.2 in [12]), the underlying reasons for this should be that (i) the 2-d energy of this 2-d UV finite QFT on a compact space (cylinder) should admit a regular inverse-mass expansion, and (ii) the space-time supersymmetry of the superstring action “spontaneously” broken by the solution should imply some kind of “asymptotic supersymmetry”. That would mean that in the limit when  $J$  is sent to infinity for fixed  $\frac{\lambda}{J^2}$  the classical expression for the ground-state energy (5.15), (5.22) and the energies of excited string states obtained from quadratic fluctuations are *exact*, just like in the BMN case.

Similar conclusions should apply for all multispin string solutions that have energy admitting a regular expansion in  $\frac{\lambda}{J^2}$  as in (1.2). If there is indeed a general relation between the regularity of the classical expression of the energy and suppression of quantum corrections to it in the  $J \rightarrow \infty$  limit remains to be understood.

As discussed in section 1, it should be then be possible to compare the classical energy with the SYM anomalous dimension also computed in the limit of large  $J$ . Such

a comparison was indeed successfully performed for the 2-spin circular [16, 18, 19] and folded [16, 17, 19] string solutions and the 3-spin circular solution of [14] with two equal spins [21].

Another interesting open problem is to compare the string 1-loop  $\frac{1}{J}$  correction to the leading  $\frac{\lambda}{J}$  term in (5.48) with the corresponding  $\frac{1}{J}$  correction to the thermodynamic limit of the Bethe ansatz expressions for the anomalous dimension [21] on the gauge theory side.

## 6 Rotating strings in $AdS_5 \times S^5$

Let us now generalize the discussion of section 4 to the case when the string can rotate in both  $AdS_5$  and  $S^5$ . For that we need to supplement the  $S^5$  rotating string ansatz (4.2) by a similar  $AdS_5$  one [14, 18, 35]:

$$\begin{aligned} Y_0 &\equiv Y_5 + iY_0 = z_0(\sigma)e^{i\omega_0\tau} , \\ Y_1 &\equiv Y_1 + iY_2 = z_1(\sigma)e^{i\omega_1\tau} , \quad Y_2 \equiv Y_3 + iY_4 = z_2(\sigma)e^{i\omega_2\tau} . \end{aligned} \quad (6.1)$$

Here the functions  $z_r = (z_0, z_1, z_2)$  are, in general, complex, and because of the condition  $\eta_{MN}Y^MY^N = -1$ , their real radial parts lie on a hyperboloid ( $\eta_{rs} = (-1, 1, 1)$ , cf. (4.3))

$$z_r = r_r e^{i\beta_r} , \quad \eta^{rs} r_r r_s \equiv -r_0^2 + r_1^2 + r_2^2 = -1 . \quad (6.2)$$

In sections 4 and 5 we had  $r_0 = 1$ ,  $r_1 = r_2 = 0$ ,  $\beta_r = 0$ . To satisfy the closed string periodicity conditions we need, as in (4.4),

$$r_r(\sigma + 2\pi) = r_r(\sigma) , \quad \beta_r(\sigma + 2\pi) = \beta_r(\sigma) + 2\pi k_r , \quad (6.3)$$

where  $k_r$  are integers. Comparing (6.1) to (2.15) we conclude that the  $AdS_5$  time  $t$  and the angular coordinates  $\phi_1, \phi_2$  are related to  $\beta_r$  by

$$t = \omega_0\tau + \beta_0(\sigma) , \quad \phi_1 = \omega_1\tau + \beta_1(\sigma) , \quad \phi_2 = \omega_2\tau + \beta_2(\sigma) . \quad (6.4)$$

We shall require the time coordinate  $t$  to be single-valued, i.e. ignore windings in time direction and will also rename  $\omega_0$  into  $\kappa$ , i.e.

$$k_0 = 0 , \quad \omega_0 \equiv \kappa . \quad (6.5)$$

The three  $O(2,4)$  Cartan generators (spins) in (2.18) here are ( $S_0 = E$ ,  $\omega_r = (\omega_0, \omega_1, \omega_2)$ )

$$S_r = \sqrt{\lambda}\omega_r \int_0^{2\pi} \frac{d\sigma}{2\pi} r_r^2(\sigma) \equiv \sqrt{\lambda} \mathcal{S}_r . \quad (6.6)$$

In view of (6.2), they satisfy the relation

$$\sum_{s,r} \eta^{sr} \frac{\mathcal{S}_r}{\omega_s} = -1 , \quad \text{i.e.} \quad \frac{\mathcal{E}}{\kappa} - \frac{\mathcal{S}_1}{\omega_1} - \frac{\mathcal{S}_2}{\omega_2} = 1 . \quad (6.7)$$

Substituting the above rotational ansatz into the  $AdS_5$  Lagrangian (and changing overall sign) we find the analog of the 1-d Lagrangian (4.11) in the  $S^5$  case [35] (we assume sum over repeated indices  $r, s$ )

$$\tilde{L} = \frac{1}{2}\eta^{rs}(z'_r z'^*_s - \omega_r^2 z_r z_s^*) - \frac{1}{2}\tilde{\Lambda}(\eta^{rs} z_r z_s^* + 1) . \quad (6.8)$$

Like its  $S^5$  counterpart (4.11), this 1-d Lagrangian is a special case of an  $n = 6$  Neumann system, now with signature  $(- + + + -)$ , and thus represents again an integrable system, being related to a special euclidean-signature Neumann model by an analytic continuation. The reduction of the total  $AdS_5 \times S^5$  Lagrangian on the rotation ansatz is then given by the sum of (4.11) and (6.8). From (6.8) we find as in (4.16)

$$\beta'_r = \frac{u_r}{r_r^2} , \quad u_r = \text{const} , \quad (6.9)$$

so that the effective Lagrangian for the radial coordinates becomes

$$\tilde{L} = \frac{1}{2}\eta^{rs}(r'_r r'_s - \omega_r^2 r_s r_s - \frac{u_r u_s}{r_r r_s}) - \frac{1}{2}\tilde{\Lambda}(\eta^{rs} r_r r_s + 1) . \quad (6.10)$$

Thus (6.10) describes a Neumann-Rosochatius integrable system with indefinite signature, i.e. with  $\delta_{ij}$  replaced by  $\eta_{rs}$  (cf. (4.17)).

We should also require the periodicity condition analogous to (4.21):  $u_r \int_0^{2\pi} \frac{d\sigma}{r_r^2(\sigma)} = 2\pi k_r$ . Then  $k_0$  implies that we should set  $u_0 = 0$  as a consequence of single-valuedness of the  $AdS_5$  time.

While the equations for  $r_i$  and  $r_r$  following from (4.15) and (6.10) are decoupled, the variables of the two NR systems are mixed in the conformal gauge constraints (2.7),(2.8) which now take the form (generalizing (4.19),(4.20) where we had  $r_0 = 1$ ,  $u_r = 0$ ,  $r_a = 0$ )

$$r_0'^2 + \kappa^2 r_0^2 = \sum_{a=1}^2 (r_a'^2 + \omega_a^2 r_a^2 + \frac{u_a^2}{r_a^2}) + \sum_{i=1}^3 (r_i'^2 + w_i^2 r_i^2 + \frac{v_i^2}{r_i^2}) , \quad (6.11)$$

$$\sum_{a=1}^2 \omega_a u_a + \sum_{i=1}^3 w_i v_i = 0 . \quad (6.12)$$

Here  $r_0^2 - \sum_{a=1}^2 r_a^2 = 1$ , and  $\sum_{i=1}^3 r_i^2 = 1$  and we used that  $u_0 = 0$ . One can then repeat the discussion of sections 4.2, 4.3 and 4.4 in the present case, classifying general solutions of the resulting NR system. One again finds folded and circular solutions, and the two-spin folded solution exists only if the string is bent [18].

## 6.1 Simple circular strings in $AdS_5$

Let us first assume that the string is not rotating in  $S^5$  (i.e.  $w_i, v_i = 0$ ,  $r_i = \text{const}$ ) and consider the  $AdS_5$  analog of the simplest circular solution of section 5 by demanding

$\tilde{\Lambda} = \text{const}$ . The discussion is exactly the same as in section 5 with few signs reversed. As in section 5.1, finding solutions with  $\tilde{\Lambda} = \text{const}$  turns out to be equivalent to looking for constant radii ( $r_r = \text{const}$ ) solutions. Then (cf. (5.7),(5.8))

$$r_r = a_r = \text{const} , \quad \beta_a = k_a \sigma , \quad k_0 = 0 , \quad u_0 = 0 , \quad u_a = a_a^2 k_a , \quad (6.13)$$

$$\omega_0^2 \equiv \kappa^2 = \tilde{\Lambda} , \quad \omega_a^2 = k_a^2 + \kappa^2 , \quad a = 1, 2 . \quad (6.14)$$

The energy as a function of spins is then obtained by solving the system that follows from the definition of the charges (6.6) and the constraints (6.11),(6.12) with  $\kappa$  as a parameter (cf. (5.10)–(5.11))

$$\frac{\mathcal{E}}{\kappa} - \frac{\mathcal{S}_1}{\sqrt{k_1^2 + \kappa^2}} - \frac{\mathcal{S}_2}{\sqrt{k_2^2 + \kappa^2}} = 1 , \quad (6.15)$$

$$\kappa \mathcal{E} - \frac{1}{2} \kappa^2 = \sqrt{k_1^2 + \kappa^2} \mathcal{S}_1 + \sqrt{k_2^2 + \kappa^2} \mathcal{S}_2 , \quad k_1 \mathcal{S}_1 + k_2 \mathcal{S}_2 = 0 . \quad (6.16)$$

This implies  $\frac{\mathcal{S}_1 k_1^2}{\sqrt{k_1^2 + \kappa^2}} + \frac{\mathcal{S}_2 k_2^2}{\sqrt{k_2^2 + \kappa^2}} = \frac{1}{2} \kappa^2$ . Considering the limit of large spins  $\mathcal{S}_i \gg 1$ , with  $k_a$  being fixed we conclude that  $\kappa = (2k_1^2 \mathcal{S}_1 + 2k_2^2 \mathcal{S}_2)^{1/3} + \dots$  and then

$$\mathcal{E} = \mathcal{S}_1 + \mathcal{S}_2 + \frac{3}{4} (2k_1^2 \mathcal{S}_1 + 2k_2^2 \mathcal{S}_2)^{1/3} + \dots , \quad (6.17)$$

or, in view of  $k_1 \mathcal{S}_1 = -k_2 \mathcal{S}_2$  (treating  $\mathcal{S}_1, \mathcal{S}_2$  and  $k_1$  as independent)

$$\mathcal{E} = \mathcal{S} + \frac{3}{4} (2k_1^2 \mathcal{S} \frac{\mathcal{S}_1}{\mathcal{S}_2})^{1/3} + \dots , \quad \mathcal{S} \equiv \mathcal{S}_1 + \mathcal{S}_2 . \quad (6.18)$$

Using (6.6) this can be rewritten as

$$E = S + \frac{3}{4} (\lambda S)^{1/3} (2k_1^2 \frac{\mathcal{S}_1}{\mathcal{S}_2})^{1/3} + \dots . \quad (6.19)$$

The case of  $k_1 = -k_2 = k$  when the two spins are equal  $\mathcal{S}_1 = \mathcal{S}_2 = \frac{1}{2} \mathcal{S}$  is that of the the circular solution found in [14] for which we get

$$E = S + \frac{3}{4} (2k^2 \lambda S)^{1/3} + \dots . \quad (6.20)$$

As was shown in [14], this  $k_1 = -k_2$  solution is stable only for small enough  $\mathcal{S}$  (namely,  $\mathcal{S} \leq \frac{5}{8} \sqrt{\frac{7}{2}}$  for  $k = 1$ ).

The “non-perturbative” scaling of the subleading term in (6.19) with  $\lambda$  precludes one from direct comparison of the above energies to anomalous dimensions of the corresponding [14] SYM operators which (in euclidean version) have the following structure [14]  $\text{tr}(\Phi(D_1 + iD_2)^{S_1}(D_3 + iD_4)^{S_2}\Phi) + \dots$ . This is unfortunate, since such operators are of more “realistic” type similar to the ones relevant for high-energy scattering in non-supersymmetric gauge theories – they contain many covariant derivatives

instead of many scalars and thus may appear in less supersymmetric gauge theories without adjoint scalars.

It turns out that one needs a large  $J$  spin in  $S^5$  directions to have a regular (1.2) expansion of the energy. Indeed, the situation changes when we consider “hybrid” solutions where the circular string rotates in both  $AdS_5$  and  $S_5$  directions.

## 6.2 Constant radii circular strings in $AdS_5 \times S^5$

It is straightforward to combine the solutions of sections 5.1 and 6.1 to write down the most general circular constant-radii solution in  $AdS_5 \times S^5$  [35]. It is parametrized by the frequencies ( $a = 1, 2$ ;  $i = 1, 2, 3$ )

$$\omega_0 = \kappa , \quad \omega_a^2 = k_a^2 + \kappa^2 , \quad w_i^2 = m_i^2 + \nu^2 , \quad \kappa^2 = \tilde{\Lambda} , \quad \nu^2 = -\Lambda , \quad (6.21)$$

related to the energy  $\mathcal{E}$  and 2+3 spins  $\mathcal{S}_a$  and  $\mathcal{J}_i$  and topological numbers  $k_a$  and  $m_i$ . These will be related by (5.10) and (6.7) as well as by the conformal gauge constraints (6.11) and (6.12). Explicitly, we get the following generalization of both (5.9)–(5.11) and (6.15),(6.16)

$$\sum_{i=1}^3 \frac{\mathcal{J}_i}{\sqrt{m_i^2 + \nu^2}} = 1 , \quad \frac{\mathcal{E}}{\kappa} - \sum_{a=1}^2 \frac{\mathcal{S}_a}{\sqrt{k_a^2 + \kappa^2}} = 1 , \quad (6.22)$$

$$2\kappa\mathcal{E} - 2 \sum_{a=1}^2 \sqrt{k_a^2 + \kappa^2} \mathcal{S}_a - \kappa^2 = 2 \sum_{i=1}^3 \sqrt{m_i^2 + \nu^2} \mathcal{J}_i - \nu^2 , \quad (6.23)$$

$$\sum_{a=1}^2 k_a \mathcal{S}_a + \sum_{i=1}^3 m_i \mathcal{J}_i = 0 . \quad (6.24)$$

Here  $\kappa$  and  $\nu$  (or the two Lagrange multipliers in (6.21)) are parameters that need to be solved for in order to find  $\mathcal{E}$  as a function of the spins  $\mathcal{S}_a, \mathcal{J}_i$  and windings  $k_a, m_i$ . The solution exists only for such integers  $k_a$  and  $m_i$  that satisfy (6.24).

If all spins are of the same order and large  $\mathcal{S}_a \sim \mathcal{J}_i \gg 1$  we find

$$\begin{aligned} \kappa &= \mathcal{J} + \frac{1}{2\mathcal{J}^2} \left( \sum_{i=1}^3 m_i^2 \mathcal{J}_i + 2 \sum_{a=1}^2 k_a^2 \mathcal{S}_a \right) + O\left(\frac{1}{\mathcal{J}^2}\right) , & \mathcal{J} &\equiv \sum_{i=1}^3 \mathcal{J}_i , \\ \nu &= \mathcal{J} - \frac{1}{2\mathcal{J}^2} \sum_{i=1}^3 m_i^2 \mathcal{J}_i + O\left(\frac{1}{\mathcal{J}^2}\right) , \end{aligned} \quad (6.25)$$

and thus ( $\mathcal{S} \equiv \sum_{a=1}^2 \mathcal{S}_a$ )

$$\mathcal{E} = \mathcal{J} + \mathcal{S} + \frac{1}{2\mathcal{J}^2} \left( \sum_{i=1}^3 m_i^2 \mathcal{J}_i + \sum_{a=1}^2 k_a^2 \mathcal{S}_a \right) + O\left(\frac{1}{\mathcal{J}^3}\right) , \quad (6.26)$$

or [35]

$$E = J + S + \frac{\lambda}{2J^2} \left( \sum_{i=1}^3 m_i^2 \mathcal{J}_i + \sum_{a=1}^2 k_a^2 \mathcal{S}_a \right) + O\left(\frac{\lambda^2}{J^3}\right) . \quad (6.27)$$



This expression is a direct generalization of (5.15) in the  $\mathcal{S}_a = 0$  case. The energy is minimal if  $m_i^2$  and  $k_a^2$  have minimal possible values (0 or 1). We may also look at a different limit when  $\mathcal{J} \gg \mathcal{S} \gg 1$ . In this case we get a “BMN-type” (single  $J$  rotation) asymptotics with the leading term still given by (6.27), i.e.  $\Delta E \sim \frac{\lambda}{2J^2} S$ .

As an example, let us consider the simplest hybrid solution when only one of each types of the spin is non-zero, i.e.  $\mathcal{J}_1 = \mathcal{J}$ ,  $\mathcal{S}_1 = \mathcal{S}$ ,  $\mathcal{S}_2 = \mathcal{J}_2 = \mathcal{J}_3 = 0$ . Then  $r_0^2 - r_1^2 = 1$ ,  $r_3 = 0$  and  $r_1 = 1$ ,  $r_2 = r_3 = 0$ , i.e. (cf. (2.15))

$$Y_0 = \cosh \rho_0 e^{i\kappa\tau}, \quad Y_1 = \sinh \rho_0 e^{i\omega\tau + ik\sigma}, \quad X_1 = e^{iw\tau + im\sigma}, \quad (6.28)$$

where  $r_0 = \cosh \rho_0$  determines the fixed radial coordinate in  $AdS_5$  at which the string is located while it is spread and rotating in  $\phi_1$  (it is positioned at  $\theta = \frac{\pi}{2}$  and  $\phi_2 = 0$  in  $S^3$  of  $AdS_5$ ). Also, the string is a rotating circle along  $\varphi_1$  in  $S^5$  located at  $\varphi_2 = \varphi_3 = 0$ ,  $\gamma = \frac{\pi}{2}$ ,  $\psi = 0$ . Its energy for  $\mathcal{J} \sim \mathcal{S} \gg 1$  is then

$$E = J + S + \frac{\lambda}{2J^2}(m^2 J + k^2 S) + \dots \quad (6.29)$$

One can easily analyse the fluctuations near this solution as was done in in section 5.3 [35]. We find 1 massless and 4 massive (mass  $\nu$ ) fluctuations in  $S^5$  directions; in addition to 2 massive (mass  $\kappa$ ) decoupled  $AdS_5$  fluctuations there are also 3 coupled ones with a Lagrangian similar to (5.34). Then the equation (5.36) for the characteristic frequencies becomes

$$(\Omega^2 - n^2)^2 + 4r_1^2(\kappa\Omega)^2 - 4r_0^2(\omega_1\Omega - kn)^2 = 0, \quad (6.30)$$

and one concludes that this  $(S, J)$  solution is always stable. Indeed, setting  $r_0 = a$ ,  $r_1 = \sqrt{a^2 - 1}$  we get

$$\Omega_- = \frac{1}{2\kappa} n \left[ 2a^2 k \pm \sqrt{n^2 + 4a^2(a^2 - 1)k^2} \right] + O\left(\frac{1}{\kappa^3}\right), \quad (6.31)$$

so that for any  $a = \cosh \rho_0 \geq 1$  there are no unstable modes.

The conclusion is that for a regular large-spin expansion of the energy one needs to have at least one (large) component of spin in  $S^5$  direction. This turns out to be true also in the case of other (folded and circular) spinning string solutions with more complicated  $\sigma$ -dependence.

## 7 “Inhomogeneous” two-spin solutions in $AdS_5 \times S^5$

### 7.1 Rotating ansatz in terms of angles

If we set  $k_a$  and  $m_i$  or  $u_r$  in (6.10) and  $v_i$  in (4.17) to zero (i.e. assume that the angles  $\phi_a$  and  $\varphi_i$  do not depend on  $\sigma$ ), the  $AdS_5 \times S^5$  NR system reduces to the sum of the two  $n = 3$  Neumann systems. Then rotating strings carrying 2+3 charges

$(S_1, S_2; J_1, J_2, J_3)$  and the energy  $E$  are described by the following ansatz in terms of angles in (2.15) [14]

$$t = \kappa\tau, \quad \phi_a = \omega_a\tau, \quad \varphi_i = w_i\tau, \quad \rho(\sigma) = \rho(\sigma + 2\pi). \quad (7.1)$$

The remaining angles may depend only on  $\sigma$ , i.e.  $\theta = \theta(\sigma)$ ,  $\gamma = \gamma(\sigma)$  and  $\psi = \psi(\sigma)$  and may be periodic modulo  $2\pi$  shift, e.g.,  $\psi(\sigma + 2\pi) = \psi(\sigma) + 2\pi n$ . If  $n = 0$  we get *folded* solutions, if  $n \neq 0$  we get *circular* solutions [18].

The conserved charges in (2.18),(2.19) then have the following explicit form

$$\begin{aligned} \mathcal{S}_1 &= \omega_1 \int_0^{2\pi} \frac{d\sigma}{2\pi} \sinh^2 \rho \cos^2 \theta, & \mathcal{J}_1 &= w_1 \int_0^{2\pi} \frac{d\sigma}{2\pi} \sin^2 \gamma \cos^2 \psi, \\ \mathcal{S}_2 &= \omega_2 \int_0^{2\pi} \frac{d\sigma}{2\pi} \sinh^2 \rho \sin^2 \theta, & \mathcal{J}_2 &= w_2 \int_0^{2\pi} \frac{d\sigma}{2\pi} \sin^2 \gamma \sin^2 \psi, \\ \mathcal{E} &= \kappa \int_0^{2\pi} \frac{d\sigma}{2\pi} \cosh^2 \rho, & \mathcal{J}_3 &= w_3 \int_0^{2\pi} \frac{d\sigma}{2\pi} \cos^2 \gamma. \end{aligned} \quad (7.2)$$

The sigma model equations for the  $\sigma$ -dependent angles  $(\rho, \theta)$

$$\begin{aligned} \rho'' - \sinh \rho \cosh \rho (\kappa^2 + \theta'^2 - \omega_1^2 \cos^2 \theta - \omega_2^2 \sin^2 \theta) &= 0, \\ (\sinh^2 \rho \theta')' - (\omega_1^2 - \omega_2^2) \sinh^2 \rho \sin \theta \cos \theta &= 0, \end{aligned} \quad (7.3)$$

and  $(\gamma, \psi)$

$$\begin{aligned} \gamma'' - \sin \gamma \cos \gamma (w_3^2 + \psi'^2 - w_1^2 \cos^2 \psi - w_2^2 \sin^2 \psi) &= 0, \\ (\sin^2 \gamma \psi')' - (w_1^2 - w_2^2) \sin^2 \gamma \sin \psi \cos \psi &= 0, \end{aligned} \quad (7.4)$$

are decoupled from each other. As explained above and in [18], the resulting system of equations is completely integrable, being equivalent to a combination of the two Neumann dynamical systems. As a result, there are 2+2 “hidden” integrals of motion, reducing the general problem to solution of two independent systems of two coupled first-order equations, with parameters related through the only one nontrivial conformal gauge constraint

$$\begin{aligned} \rho'^2 - \kappa^2 \cosh^2 \rho + \sinh^2 \rho (\theta'^2 + \omega_1^2 \cos^2 \theta + \omega_2^2 \sin^2 \theta) \\ + \gamma'^2 + w_3^2 \cos^2 \gamma + \sin^2 \gamma (\psi'^2 + w_1^2 \cos^2 \psi + w_2^2 \sin^2 \psi) &= 0. \end{aligned} \quad (7.5)$$

Note that the two metrics in (2.16),(2.17) are related by the obvious analytic continuation and change of the overall sign, which is equivalent, for the present rotational ansatz (7.1), to

$$\rho \leftrightarrow i\gamma, \quad \theta \leftrightarrow \psi, \quad \kappa \leftrightarrow -w_3, \quad \omega_1 \leftrightarrow -w_1, \quad \omega_2 \leftrightarrow -w_2. \quad (7.6)$$

This transformation maps the system (7.3) into the system (7.4) and also preserves the constraint (7.5). Thus it formally maps solutions into solutions [18, 19]. Under (7.6) the conserved charges (7.2) transform as follows

$$S_1 \leftrightarrow J_1, \quad S_2 \leftrightarrow J_2, \quad E \leftrightarrow -J_3. \quad (7.7)$$

This corresponds to interchanging different  $SO(2)$  generators of the symmetry group  $SO(2,4) \times SO(6)$ . One can find also other similar transformations that map solutions into solutions by combining (7.6) with discrete  $SO(2,4) \times SO(6)$  isometries like interchanging the angular coordinates (see below).

## 7.2 Folded two-spin solutions: $(S, J)$ and $(J_1, J_2)$

Let us now review the two non-trivial two-spin folded string solutions which are, in fact, related by the above analytic continuation.

The first is the “ $(S, J)$ ” solution [11]

$$\kappa, \omega_1, w_3 \neq 0, \quad \rho = \rho(\sigma), \quad \theta = 0, \quad \gamma = 0, \quad \psi = 0, \quad (7.8)$$

where the string is stretched in the radial direction  $\rho$  of  $AdS_5$  and rotates ( $\omega_1$ ) in  $AdS_5$  about its center of mass. The latter in turn moves ( $w_3$ ) along a large circle of  $S^5$ . In the limit of point-like string ( $\mathcal{S} = 0$ ) this becomes a massless geodesic in  $S^5$  as in [4, 8]. In the case of  $w_3 = 0$  this becomes the folded string rotating in  $AdS_5$  [37, 8]. The gauge constraint (7.5) and the integrals of motion (7.2) here become

$$\begin{aligned} \rho'^2 - \kappa^2 \cosh^2 \rho + \omega_1^2 \sinh^2 \rho &= -w_3^2, & \mathcal{J} &\equiv \mathcal{J}_3 = w_3, \\ \mathcal{S} \equiv \mathcal{S}_1 &= \omega_1 \int_0^{2\pi} \frac{d\sigma}{2\pi} \sinh^2 \rho, & \mathcal{E} &= \kappa \int_0^{2\pi} \frac{d\sigma}{2\pi} \cosh^2 \rho. \end{aligned} \quad (7.9)$$

For the second “ $(J_1, J_2)$ ” solution one has [14, 17]

$$\kappa, w_1, w_2 \neq 0, \quad \rho = 0, \quad \theta = 0, \quad \gamma = \frac{\pi}{2}, \quad \psi = \psi(\sigma). \quad (7.10)$$

Here the string is located at the center of  $AdS_5$  while it is stretched ( $\psi$ ) along a great circle of  $S^5$  and rotates ( $w_2$ ) about its center of mass which moves ( $w_1$ ) along an orthogonal great circle of  $S^5$ . The gauge constraint (7.5) and the integrals of motion (7.2) here are

$$\begin{aligned} \psi'^2 + w_1^2 \cos^2 \psi + w_2^2 \sin^2 \psi &= \kappa^2, & \mathcal{E} &= \kappa, \\ \mathcal{J}_1 &= w_1 \int_0^{2\pi} \frac{d\sigma}{2\pi} \cos^2 \psi, & \mathcal{J}_2 &= w_2 \int_0^{2\pi} \frac{d\sigma}{2\pi} \sin^2 \psi. \end{aligned} \quad (7.11)$$

In view of (7.6),(7.7) we conclude that these two solutions are related by the following analytic continuation:

$$\begin{aligned} \rho &\rightarrow i\psi, & \kappa &\rightarrow -w_1, & \omega_1 &\rightarrow -w_2, & w_3 &\rightarrow -\kappa, \\ E &\rightarrow -J_1, & S &\rightarrow J_2, & J &\rightarrow -E. \end{aligned} \quad (7.12)$$

Here we are assuming that  $\rho(\sigma + 2\pi) = \rho(\sigma)$  and  $\psi(\sigma + 2\pi) = \psi(\sigma)$ , i.e.  $\psi$  does not have a winding number. This choice corresponds to a folded string solution (a two-spin generalization of the solution of [37, 8]).

The first-order equations in (7.9) and (7.11) are first integrals of the sinh-Gordon and sine-Gordon equations for  $\rho$  and  $\psi$ , respectively, which are the only non-trivial equations of the total Neumann systems one is to solve in the present two cases: here the related hyperelliptic curve (see section 4.3) reduces to an elliptic one. Indeed, their solutions can be readily expressed in terms of elliptic functions (see below).

One can also directly relate [19] the systems of equations expressing the periodicity condition and the respective energies and spins. In the first case [11] we get, introducing a modular parameter  $q < 0$  related to the maximal value of the radial  $AdS_5$  coordinate  $\rho_0$

$$q \equiv -\sinh^2 \rho_0 = \frac{\kappa^2 - w_3^2}{\kappa^2 - \omega_1^2} < 0, \quad (7.13)$$

$$\sqrt{\kappa^2 - w_3^2} = \frac{2\sqrt{-q}}{\pi} K(q), \quad \mathcal{E} = \kappa + \frac{\kappa}{\omega_1} \mathcal{S} = \frac{2\kappa\sqrt{-q}}{\pi\sqrt{\kappa^2 - w_3^2}} E(q),$$

where  $K(q)$  and  $E(q)$  are the standard complete elliptic integrals of the first and the second kind.\* Solving for  $\omega_1$  and  $\kappa$  in terms of  $\mathcal{J}$  and  $q$  we get the system of two equations for the energy as a function of the spins  $\mathcal{E} = \mathcal{E}(\mathcal{S}, \mathcal{J})$  [19]

$$\left(\frac{\mathcal{J}}{K(q)}\right)^2 - \left(\frac{\mathcal{E}}{E(q)}\right)^2 = \frac{4}{\pi^2} q, \quad (7.14)$$

$$\left(\frac{\mathcal{S}}{K(q) - E(q)}\right)^2 - \left(\frac{\mathcal{J}}{K(q)}\right)^2 = \frac{4}{\pi^2} (1 - q), \quad (7.15)$$

where the parameters  $q$  is negative for a physical folded solution. The second of these two parametric equations determines  $q$  in terms of  $\mathcal{S}$  and  $\mathcal{J}$ , while the first one then gives the energy as a function of the spins.

Similarly, for the  $(J_1, J_2)$  solution (7.10) one finds [17] (we assume  $\omega_2^2 > \omega_1^2$ ; here  $\psi_0$  is the maximal value of  $\psi$ )

$$q \equiv \sin^2 \psi_0 = \frac{\kappa^2 - w_1^2}{w_2^2 - w_1^2} > 0, \quad 1 = \frac{\mathcal{J}_1}{w_1} + \frac{\mathcal{J}_2}{w_2}, \quad \mathcal{E} = \kappa,$$

$$\mathcal{J}_1 = \frac{2w_1}{\pi\sqrt{w_2^2 - w_1^2}} E(q), \quad \sqrt{w_2^2 - w_1^2} = \frac{2}{\pi} K(q). \quad (7.16)$$

The solution of the equation in (7.11) for  $\psi$  can be written as follows

$$\cos \psi(\sigma) = r_1(\sigma) = \text{dn}(A\sigma, q), \quad \sin \psi(\sigma) = r_2(\sigma) = \sqrt{q} \text{sn}(A\sigma, q), \quad (7.17)$$

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\*They are defined by  $K(q) = \int_0^{\frac{\pi}{2}} \frac{d\alpha}{\sqrt{1-q\sin^2\alpha}}$  and  $E(q) = \int_0^{\frac{\pi}{2}} d\alpha \sqrt{1-q\sin^2\alpha}$  and are related to hypergeometric functions  ${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1, q) = \frac{2}{\pi} K(q)$ ,  ${}_2F_1(-\frac{1}{2}, \frac{1}{2}; 1, q) = \frac{2}{\pi} E(q)$ . Let us note also that the elliptic integral of the third kind is defined by  $\Pi(m^2, q) = \int_0^{\pi/2} \frac{d\alpha}{(1-m^2\sin^2\alpha)\sqrt{1-q\sin^2\alpha}}$ .

where  $r_3(\sigma) = 0$  ( $\gamma = \frac{\pi}{2}$ , cf.(2.15)),  $A \equiv \frac{2}{\pi}K(q)$  and  $\text{dn}$  and  $\text{sn}$  are the standard elliptic functions.<sup>†</sup> Here we finish with the system of two equations determining  $\mathcal{E} = \mathcal{E}(\mathcal{J}_1, \mathcal{J}_2)$

$$\left(\frac{\mathcal{E}}{K(q)}\right)^2 - \left(\frac{\mathcal{J}_1}{E(q)}\right)^2 = \frac{4}{\pi^2}q, \quad (7.18)$$

$$\left(\frac{\mathcal{J}_2}{K(q) - E(q)}\right)^2 - \left(\frac{\mathcal{J}_1}{E(q)}\right)^2 = \frac{4}{\pi^2}, \quad (7.19)$$

where  $q > 0$ . A manifestation of the analytic continuation relation (7.12) between the two solutions is the equivalence [19] of the systems (7.14),(7.15) and (7.18),(7.19) under the substitution

$$\mathcal{E} \mapsto -\mathcal{J}_1, \quad \mathcal{S} \mapsto \mathcal{J}_2, \quad \mathcal{J} \mapsto -\mathcal{E}, \quad (7.20)$$

and the analytic continuation from  $q > 0$  to  $q < 0$  in the elliptic integrals.

### 7.3 Energy as function of spins

Depending on a region of parameter space (or values of the integrals of motion) one finds different functional form of dependence of the energy on the two spins. A direct comparison with gauge theory is possible in the case when the two spins are large compared to  $\sqrt{\lambda}$ , i.e.  $\mathcal{S} \gg 1$ ,  $\mathcal{J} \gg 1$  in the  $(S, J)$  case and  $\mathcal{J}_1 \gg 1$ ,  $\mathcal{J}_2 \gg 1$  in the  $(J_1, J_2)$  case. We can then expand the energies, e.g., in powers of the total  $S^5$  spin  $\mathcal{J}$ . This amounts to an expansion in (inverse) powers of  $\mathcal{J} \equiv \mathcal{J}_3 = \frac{J_3}{\sqrt{\lambda}}$  in the  $(S, J)$  case and of  $\mathcal{J} \equiv \mathcal{J}_1 + \mathcal{J}_2 = \frac{1}{\sqrt{\lambda}}(J_1 + J_2)$  in the  $(J_1, J_2)$  case, respectively,

$$E = S + J + \frac{\lambda}{J} \tilde{\epsilon}_1\left(\frac{S}{J}\right) + \frac{\lambda^2}{J^3} \tilde{\epsilon}_2\left(\frac{S}{J}\right) + \dots, \quad J \equiv J_3, \quad J, S \gg \sqrt{\lambda}, \quad (7.21)$$

$$E = J + \frac{\lambda}{J} \epsilon_1\left(\frac{J_2}{J}\right) + \frac{\lambda^2}{J^3} \epsilon_2\left(\frac{J_2}{J}\right) + \dots, \quad J \equiv J_1 + J_2, \quad J_1, J_2 \gg \sqrt{\lambda}. \quad (7.22)$$

The coefficient functions  $\tilde{\epsilon}_n$  and  $\epsilon_n$  in (7.21) and (7.22) (the analogs of  $c_n$  in (1.2)) can be related, given that the two solutions are related by the analytic continuation (7.20). By expanding in large spins one finds a simple relation between the leading-order (“one-loop”) corrections for the energies of the two solutions [19]:

$$\tilde{\epsilon}_1(y) = -\epsilon_1(-y). \quad (7.23)$$

The same relation is obtained also on the gauge-theory side [19].

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<sup>†</sup>The Jacobi elliptic function  $\text{sn}(u, q)$  is defined by  $u = \int_0^{\text{sn}(u, q)} \frac{dy}{\sqrt{(1-y^2)(1-ky^2)}}$ . Equivalently, if  $\text{sn } u = \sin \phi$  then  $u = \int_0^\phi \frac{d\alpha}{\sqrt{1-k^2 \sin^2 \alpha}}$ . One has also  $\text{dn}^2(u, q) + q \text{sn}^2(u, q) = 1$  and  $\text{cn}^2(u, q) + \text{sn}^2(u, q) = 1$ . These three functions (given by ratios of theta-functions) are meromorphic. Also,  $\text{sn}(-u, q) = -\text{sn}(u, q)$  and  $\text{sn}(u + 2I, q) = -\text{sn}(u, q)$ , where the half-period is  $I = K(q)$ .

Eq. (7.23) follows also directly from (7.14) and (7.16) or the systems (7.14),(7.15) and (7.18),(7.19). In the  $(J_1, J_2)$  case, expanding the parameter  $q$  for large  $\mathcal{J}$  as (with  $\mathcal{J}$  being  $\mathcal{J}_1 + \mathcal{J}_2$ )

$$q = q_0 + \frac{q_1}{\mathcal{J}^2} + \frac{q_2}{\mathcal{J}^4} + \dots, \quad (7.24)$$

one finds that  $q_0$  is given by the solution of the transcendental equation

$$\frac{E(q_0)}{K(q_0)} = 1 - \frac{J_2}{J}, \quad q_0 = q_0\left(\frac{J_2}{J}\right). \quad (7.25)$$

The rest of the expansion coefficients in  $q$  and the energy (7.22) are then determined simply by linear algebra. In particular, one finds [19]

$$\epsilon_1 = \frac{2}{\pi^2} K(q_0)[E(q_0) - (1 - q_0)K(q_0)]. \quad (7.26)$$

In the  $(S, J)$  case, using the same expansion (7.24) for the corresponding parameter  $q$  in (7.13) where now  $\mathcal{J} = \mathcal{J}_3$  we find that  $q_0$  satisfies

$$\frac{E(q_0)}{K(q_0)} = 1 + \frac{S}{J}, \quad q_0 = q_0\left(\frac{S}{J}\right), \quad (7.27)$$

and also

$$\tilde{\epsilon}_1 = -\frac{2}{\pi^2} K(q_0)[E(q_0) - (1 - q_0)K(q_0)]. \quad (7.28)$$

Comparing (7.25),(7.26) to (7.27),(7.28) and observing that to the leading order (7.12) implies  $J_2 \rightarrow S$ ,  $J \rightarrow -J$ , we indeed observe the relation (7.23), or  $(\tilde{\epsilon}_1)_{q_0} = -(\epsilon_1)_{-q_0}$ .

Let us now comment on dependence of the energy on the spins in other regions of the parameter space. Let us start with the  $(S, J)$  solution. In the limit of short strings with  $\mathcal{J} \ll 1$ ,  $\mathcal{S} \ll 1$  one finds [11]

$$E = \sqrt{J^2 + 2\sqrt{\lambda} S} + \dots. \quad (7.29)$$

This limit probes a small-curvature region of  $AdS_5$  where  $\rho \approx 0$ , and where the energy spectrum should thus be approximately the same as in flat space. Indeed, (7.29) is the standard relativistic expression for the energy of a string in flat space moving with momentum  $\mathcal{J}$  and rotating in a 2-plane with spin  $\mathcal{S}$ . If the boost energy is smaller than the rotation one, i.e. if  $\mathcal{J}^2 \ll \mathcal{S}$ , then we get the flat-space Regge trajectory relation  $E \approx \sqrt{2\sqrt{\lambda} S} + \frac{J^2}{2\sqrt{2\sqrt{\lambda} S}}$ . Such expression (with a non-analytic dependence on  $\lambda$ ) cannot be directly compared to SYM theory before one computes all quantum string sigma model  $\frac{1}{\sqrt{\lambda}}$  corrections (and resums them to have a regular  $\lambda \rightarrow 0$  limit).

For short strings with  $\mathcal{J} \gg 1$  (and thus with  $\mathcal{J} \gg \mathcal{S}$ )

$$E = J + S + \frac{\lambda S}{2J^2} + \dots. \quad (7.30)$$

This corresponds to the BMN limit with  $S$  playing the role of the string excitation number [11].<sup>‡</sup> One may also consider a “near BMN” limit  $\frac{S}{J} \ll 1$  of this two-spin solution [11, 19]

$$E = J + S \sqrt{1 + \frac{\lambda}{J^2}} - \frac{\lambda S^2}{2J^3} \cdot \frac{1 + \frac{\lambda}{2J^2}}{1 + \frac{\lambda}{J^2}} + \dots, \quad S \ll J. \quad (7.31)$$

This represents the near BMN limit for a total of  $S$  excitations of the oscillation modes with  $n = \pm 1$ . Thus solving non-linear classical sigma model equations gives the same semiclassical spectrum as expanding the sigma model action near a point-like geodesic and then quantizing the small-fluctuation Lagrangian. We see that there is an overlap between the leading-order (large  $\sqrt{\lambda}$ ) quantum spectrum obtained by expanding near  $S^5$ -boosted point-like string state with *no* rotation in  $AdS_5$  and a classical spectrum obtained by expanding near a highly boosted *and* rotating string solution. This supports the suggestion [8, 11, 12] that parts of semiclassical  $AdS_5 \times S^5$  string spectrum can be captured by expanding near different classical string solutions.

Other asymptotic expressions are found when  $S$  is large. In this case the string can become very long and approach the boundary of  $AdS_5$ , i.e.  $\rho_0 \rightarrow \infty$ . For  $\mathcal{J} \ll \ln S$ ,  $S \gg 1$  one finds  $\mathcal{E} \approx \mathcal{S} + \frac{1}{\pi} \ln S + \frac{\pi \mathcal{J}^2}{2 \ln S}$ , i.e. [11]

$$E \approx S + \frac{\sqrt{\lambda}}{\pi} \ln \frac{S}{\sqrt{\lambda}} + \frac{\pi J^2}{2\sqrt{\lambda} \ln \frac{S}{\sqrt{\lambda}}}. \quad (7.32)$$

In the limit of  $J = 0$  this reduces to the remarkable  $\ln S$  behaviour found in [8] for the single-spin  $AdS_5$  rotating string solution in  $AdS_5$ . Having only large  $AdS_5$  spin thus does not lead to an analytic dependence of the energy on  $\lambda$  *and*, not surprisingly, is not enough to suppress quantum string sigma model corrections. Indeed, the 1-loop string correction shifts the coefficient of the  $\ln S$  term by a constant [11], and, in general, the classical  $\sqrt{\lambda}$  coefficient should thus be replaced by an “*interpolating*” function<sup>§</sup>

$$E = S + f(\lambda) \ln S + \dots, \quad f(\lambda)_{\lambda \gg 1} = \frac{\sqrt{\lambda}}{\pi} + a_1 + \frac{a_2}{\sqrt{\lambda}} + \dots. \quad (7.33)$$

The AdS/CFT correspondence implies that after a resummation  $f(\lambda)$  should admit a regular weak-coupling expansion  $f(\lambda)_{\lambda \ll 1} = q_1 \lambda + q_2 \lambda^2 + \dots$ , with (7.33) reproducing the anomalous dimension of the corresponding gauge-theory operators like  $\text{tr}(\bar{\Phi} D^S \Phi) + \dots$  (see [13]).

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<sup>‡</sup>The BMN case corresponds to expanding near a point-like string moving along big circle of  $S^5$ . In the limit  $J \rightarrow \infty$ ,  $\frac{\lambda}{J^2}$ =fixed one may drop all but quadratic fluctuation terms in the string action (which becomes then equivalent to the plane-wave [5] action [6] in the light-cone gauge). The energies of fluctuations above the BPS ground state  $E = J$  are then determined by the string fluctuation masses given by  $m^2 = \frac{1}{J^2} = \frac{\lambda}{J^2}$ .

<sup>§</sup>The one-loop coefficient computed in [11] is  $a_1 \approx -\frac{3}{2\pi} \ln 2$ . We use this opportunity to correct factor of 1/2 misprints in eqs. (6.6) and (6.9) in [11].



In the intermediate case where  $\ln \frac{S}{J} \ll \mathcal{J} \ll S$  we get [11]

$$E = S + J + \frac{\lambda}{2\pi^2 J} \ln^2 \frac{S}{J} + \dots, \quad (7.34)$$

In contrast to the large  $\mathcal{J}$  limit of the short string (small  $S$ ) case (7.31) here the third correction term is not related to the BMN-type spectrum: there the boost is large and string oscillations are small, while in the long-string case the spin  $S$  is always larger than the boost parameter  $\mathcal{J}$ . Eq. (7.34) appears to be analytic in  $\lambda$  and, assuming that string loop corrections to the coefficient of the  $\ln^2 \frac{S}{J}$  term are suppressed in the limit  $S \gg 1$ ,  $J \gg 1$ , one could hope to relate the  $\frac{\lambda}{2\pi^2 J} \ln^2 \frac{S}{J}$  term to the 1-loop anomalous dimension of the gauge-theory operators with large spin and large  $R$ -charge. Indeed, (7.34) may be viewed as a special case of (7.21), where  $\tilde{\epsilon}_1 \approx \frac{1}{2\pi^2} \ln^2 \frac{S}{J}$ . This asymptotics is indeed observed on the gauge theory side as a special case of the general relation between the string theory and the gauge theory results for the function  $\tilde{\epsilon}_1(\frac{S}{J})$  established in [19]. One concludes, in particular, that the coefficient of the  $\ln S$  term in the anomalous dimensions of the corresponding  $\mathcal{N} = 4$  SYM operators with large spin *and* large  $R$ -charge  $J$  is indeed suppressed also at weak 't Hooft coupling.

A similar analysis can be repeated for the  $(J_1, J_2)$  solution. The energy of a short string rotating in  $S^5$  with  $\mathcal{J}_1 \gg 1$ ,  $\mathcal{J}_1 \gg \mathcal{J}_2$  is given by a BMN type expression (cf. (7.30))

$$E = J + \frac{\lambda J_2}{2J^2} + \dots, \quad J_2 \ll J_1, \quad J = J_1 + J_2. \quad (7.35)$$

The full expression in the near BMN limit is (cf. (7.31))

$$E = J_1 + J_2 \sqrt{1 + \frac{\lambda}{J_1^2}} - \frac{\lambda J_2^2}{2J_1^3} \cdot \frac{1 + \frac{3\lambda}{2J_1^2}}{1 + \frac{\lambda}{J_1^2}} + \dots, \quad J_2 \ll J_1. \quad (7.36)$$

To compare to the BMN case we may set  $J_1 = J$  and then  $J_2$  represents the number of excitations.

Making string longer corresponds to increasing the spin  $J_2$ . For example, at  $J_1 = J_2$  we get [17]

$$E = J + c_1 \frac{\lambda}{J} + \dots, \quad c_1 \equiv \epsilon_1\left(\frac{1}{2}\right) = 0.356\dots, \quad J_2 = J_1 = \frac{1}{2}J. \quad (7.37)$$

When  $J_2 \rightarrow J = J_1 + J_2$ , i.e.  $J_1$  becomes small, the string extends over half a great  $S^5$  circle and [19]

$$E = J + \frac{2\lambda}{\pi^2 J(1 - J_2/J)} + \dots = J + \frac{2\lambda}{\pi^2 J_1} + \dots, \quad J_2 \approx J. \quad (7.38)$$

The point where  $J_2 = J$  can be viewed as a transition point: one half of the string can be unfolded to give a circular string discussed below. Alternatively, the case of

$J_1 = 0$  can be studied by starting with a single-spin folded rotating  $S^5$  solution with its center of mass at rest at a pole of  $S^5$  [8]. In this case for  $\mathcal{J} = \mathcal{J}_2 \gg 1$  one finds [8]

$$E = J + \frac{2}{\pi}\sqrt{\lambda} + \dots, \quad \frac{\sqrt{\lambda}}{J} \ll 1. \quad (7.39)$$

Like in the single-spin  $AdS_5$  case (cf.(7.32)) here the expansion of the energy is not analytic in  $\lambda$ , and one expects that quantum sigma model corrections should promote the subleading  $\sqrt{\lambda}$  term into a nontrivial function  $h(\lambda) = \frac{2}{\pi}\sqrt{\lambda} + k_1 + \frac{k_2}{\sqrt{\lambda}} + \dots$ .

## 7.4 Circular two-spin solution

In addition to the “homogeneous” circular two-spin solutions discussed in section 6 there are also different circular two-spin solutions of the Neumann system (4.17) with  $v_i = 0$  that generalize the “round circle”  $J_1 = J_2$  solution of [14] to the case of  $J_1 \neq J_2$  [18]. The circular string solution is given by the same ansatz (7.10) as for the folded string but now  $\psi(\sigma)$  is assumed to be periodic modulo  $2\pi$

$$\psi(\sigma + 2\pi) = \psi(\sigma) + 2\pi k. \quad (7.40)$$

In what follows we shall set the winding number  $k$  to be 1. In general, in spherical coordinates  $(\gamma, \psi)$  the equations of motion (7.4) describing this type of string are  $\gamma = \frac{\pi}{2}$  and  $\psi'' + \frac{1}{2}w_{21}^2 \sin 2\psi = 0$ ,  $w_{21}^2 = w_2^2 - w_1^2$ . Integrating once, we get  $\psi'^2 = w_{21}^2(q^{-1} - \sin^2 \psi)$ , where  $q$  is an integration constant. If  $q > 1$ , then  $q^{-1} = \sin^2 \psi_0$  and this solution describes a *folded* string extending from  $-\psi_0$  to  $\psi_0$ . If instead  $q < 1$ , then there is no turning point where  $\psi' = 0$ , and the solution describes a circular string extending all the way around the equator  $\gamma = \frac{\pi}{2}$  with  $\psi$  from 0 to  $2\pi$ : instead of folding back onto itself, the string wraps completely around a great circle of  $S^5$ . In the limit  $q \rightarrow 0$ , this solution approaches the circular string with  $J_1 = J_2$ . Thus the parameter  $q$  provides an interpolation between the circular and the folded string configurations. Note that after a rescaling  $\psi \rightarrow \frac{1}{2}\psi$  the equation for  $\psi$  describes a plane motion of a pendulum in a gravitational field. Clearly, the rotation of the pendulum requires more energy than the oscillatory motion and this explains why the energy of the circular string is bigger than that of the folded one.

The radial coordinates in (4.2) in the circular case are given by (cf. (7.17))

$$\cos \psi(\sigma) = r_1(\sigma) = \text{sn}(A\sigma, q), \quad \sin \psi(\sigma) = r_2(\sigma) = \text{cn}(A\sigma, q), \quad (7.41)$$

where again  $r_3 = 0$  and  $A \equiv \frac{2}{\pi}K(q)$ .

The set of equations for the energy and spins of this solution is (cf. (7.16)) [18]

$$\mathcal{J}_2 = \frac{w_2}{q} \left( 1 - \frac{E(q)}{K(q)} \right), \quad \mathcal{J}_1 = \frac{w_1}{q} \left( q - 1 + \frac{E(q)}{K(q)} \right), \quad (7.42)$$

$$\mathcal{E}^2 = w_1^2 + \frac{1}{q}(w_2^2 - w_1^2), \quad K(q) = \frac{\pi}{2} \sqrt{\frac{1}{q}(w_2^2 - w_1^2)}. \quad (7.43)$$

Solving for  $w_1, w_2$  we get a system of two equations for  $\mathcal{E} = \mathcal{E}(\mathcal{J}_1, \mathcal{J}_2)$  similar to the one in (7.18),(7.19)

$$\left(\frac{\mathcal{E}}{K(q)}\right)^2 - \left(\frac{q\mathcal{J}_1}{(1-q)K(q) - E(q)}\right)^2 = \frac{4}{\pi^2}, \quad (7.44)$$

$$\left(\frac{q\mathcal{J}_2}{K(q) - E(q)}\right)^2 - \left(\frac{q\mathcal{J}_1}{(1-q)K(q) - E(q)}\right)^2 = \frac{4}{\pi^2} q. \quad (7.45)$$

Note that the ansatz for the circular solution is symmetric under  $\mathcal{J}_1 \leftrightarrow \mathcal{J}_2$ ,<sup>¶</sup> and this symmetry may be seen by applying a modular transformation to the elliptic integrals [18, 19]:  $K(q) = \sqrt{1-q'} K(q')$ ,  $E(q) = \frac{E(q')}{\sqrt{1-q'}}$ ,  $1-q = \frac{1}{1-q'}$ .

In the limit when both spins are large we can expand  $q$  and the energy in powers of  $\frac{1}{J^2}$ , i.e.  $q = q_0 + \frac{q_1}{J^2} + \dots$ ,  $\mathcal{J} = \mathcal{J}_1 + \mathcal{J}_2$ , and (cf. (7.24),(7.25),(7.26))

$$E = J + \frac{\lambda}{J} \hat{\epsilon}_1\left(\frac{J_2}{J}\right) + \dots, \quad (7.46)$$

$$\hat{\epsilon}_1 = \frac{2}{\pi^2} K(q_0) E(q_0), \quad \frac{J_2}{J} = \frac{1}{q_0} \left[1 - \frac{E(q_0)}{K(q_0)}\right]. \quad (7.47)$$

The same relations for  $\hat{\epsilon}_1(\frac{J_2}{J})$  in (7.47) were reproduced for the corresponding 1-loop anomalous dimension on the gauge theory side [16, 18, 19].

One may also construct other similar solutions, e.g., by combining a folded string solution in  $AdS_5$  with folded or circular  $(J_1, J_2)$   $S^5$  solution. In this case the energy will be given by a system of three parametric equations involving  $\mathcal{E}, \mathcal{S}, \mathcal{J}_1, \mathcal{J}_2$ . Such  $(S, J_1, J_2)$  solutions may be related by an analytic continuation to special  $(J_1, J_2, J_3)$  solutions.

To conclude, as we have seen on the examples discussed above, to have regular (1.2) dependence of the string energy on  $\frac{\lambda}{J^2} \ll 1$  we need at least one large “center-of-mass” momentum in  $S^5$ . In such case quantum sigma model corrections are expected to be suppressed in the limit  $J \gg 1$ , and thus the classical string energy should represent the exact gauge-theory anomalous dimension computed in the limit  $J \gg 1$ , to all orders in perturbative expansion in  $\lambda$ . This was explicitly verified (for the leading “one-loop”  $\frac{\lambda}{J}$  term) for several types of such “regular” spinning string solutions [16, 18, 19, 21].

Having large spins in  $AdS_5$  only or only one large spin (with center of mass being at rest) in  $S^5$  appears to be not enough for the energy to have an expansion in even powers of  $\frac{\sqrt{\lambda}}{J}$  (note that the circular  $(S, J)$  solution of the NR system discussed in section 6.2 represents an exception from this rule, cf.(6.29)). In these latter cases quantum sigma model corrections are not expected to be suppressed in the large spin limit and thus the classical  $\sqrt{\lambda}$ -coefficients of the leading terms in the expansion of the energy should become promoted by the string quantum  $\frac{1}{(\sqrt{\lambda})^n}$  corrections to

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<sup>¶</sup>The direct limit  $J_2 = 0$  is not, however, well-defined for a circular  $(J_1, J_2)$  case.

non-trivial “interpolating” functions of  $\lambda$ . The latter should be resummed before one may try to compare to perturbative gauge-theory results. Comparing string theory to gauge theory at a *quantitative* level in such cases remains a challenge.

## 8 Open questions and generalizations

The above discussion of particular string solutions with “regular” expansion of the energy  $E$  in powers of  $\frac{\lambda}{J^2}$  raises several questions.

Since the direct comparison with gauge theory at present can be done only in the  $J \rightarrow \infty$  limit and in expansion in  $\frac{\lambda}{J^2} \ll 1$  it would be interesting to classify all possible solutions with such property (1.2) of the energy. One may also try to derive the general expression for the leading order coefficient in  $E = J + \frac{\lambda}{J}c_1 + \dots$ , i.e. for  $c_1$  as a functional on a space of such solutions. An interesting work in this direction is [43] which utilizes the observation [42] that the induced world-sheet metric of rotating strings with large  $J$  becomes degenerate, and that one can then develop a perturbative expansion near such world sheet. Deriving equations for the functional  $c_1$  (with the expressions in (6.27), (7.26), (7.28) and (7.47) as special solutions) may help to establish the correspondence with spin chain energy eigenstates in a more universal way than the presently known procedure based on association of a particular Bethe root distribution with a particular string solution [16, 19, 21].

It remains also to prove that for all solutions with “regular” expansion of the energy in  $\frac{\lambda}{J^2} \ll 1$  quantum superstring sigma model corrections are indeed suppressed by extra  $\frac{1}{J^n}$  factors as in (1.5). The underlying supersymmetry of the  $AdS_5 \times S^5$  string theory is certainly important for that conclusion, and a possible role of asymptotic supersymmetry at finite  $J$  and  $\lambda \rightarrow 0$  observed in [42] for simple  $S^5$  rotating solutions remains to be clarified. The string/gauge theory matching for a pulsating solution in [21] (when the  $\lambda \rightarrow 0$  limit does not give a BPS state) seems to indicate that suppression of quantum sigma model corrections may occur even under more general conditions.

More generally, as discussed in section 1, the full expression for the classical energy of a “regular” solution  $E = \sqrt{\lambda} \mathcal{E}(\frac{J}{\sqrt{\lambda}}, \dots)$  should be representing the exact dimensions of the corresponding gauge-theory operators computed in the large  $J$  limit. Here we assume that like the energy on the string side, the anomalous dimension on the gauge theory side should admit a regular double expansion in  $\frac{\lambda}{J^2} < 1$  and  $\frac{1}{J} \rightarrow 0$ , i.e. should have the form (1.7), (1.8). This remains to be proved in general for multi R-charge/spin SYM operators. The full expression for the classical energy should be a solution of some differential equations or an equivalent system of algebraic or transcendental equations involving moduli parameters of the string solutions (cf. (6.22)-(6.24) or (7.14), (7.15) and (7.44), (7.45), see also [35]). Thus, like in the examples discussed above, the full expression for the energy  $E$  should be effectively determined by its leading-order term  $c_1$ . That suggests a possibility to derive (in large  $J$  limit) the exact in  $\lambda$  expressions for the corresponding anomalous dimensions

in SYM theory. By analogy with how the simple square root expression  $\sqrt{1 + \frac{\lambda}{J^2} n^2}$  of the near-BPS BMN case was reproduced in [10], one may expect that in the  $J \rightarrow \infty$  limit there may then be a relation between the values of anomalous dimensions (or, in fact, the expressions for the dilatation operator restricted to a particular subsector of states) at different orders in expansion in  $\lambda$ .

Let us add that while the full expression for the classical string energy comes from the conformal gauge constraint (and looks like a “relativistic” expression  $E^2 = J^2 + 2\sqrt{\lambda} c_1 + \dots$ ) the 1-loop anomalous dimensions on the SYM side are obtained by solving the quantum spin chain Hamiltonian eigenvalue problem ( $\Delta - J = a_1 \frac{\lambda}{J} + \dots$ ), which looks like a first term in a “non-relativistic” expansion. It would be important to understand how the perturbative series on the SYM side can be summed up, i.e. how the 1-loop expression for anomalous dimension can be promoted to the full “relativistic” expression without order-by-order analysis of modification of the dilatation operator (interpreted as a generalised “non-local” spin chain Hamiltonian, cf. [45, 46]).

Another interesting problem is to compare subleading terms in  $1/J$  expansion, as was done in the BMN case in [40, 41].\* This will involve computing one-loop correction to the classical string energy (1.5) and comparing it with the subleading correction to the “thermodynamic” limit of the 1-loop Bethe energies. Note that the 1-loop (order  $\lambda$ ) SYM result for the anomalous dimension for any value of  $J$ , i.e.  $q_1(J)$  in (1.7), should represent the sum of all string sigma model loop corrections to the leading  $\frac{\lambda}{J}$  term ( $c_1$  coefficient) in (1.2). The subleading  $1/J$  terms should be governed by the same integrable structures on the two sides of the duality. For any value of  $J$ , one certainly expects that, in view of the conformal invariance of the  $AdS_5 \times S^5$  string theory (absence of mass generation), the classical integrability of the  $AdS_5 \times S^5$  sigma model [57] should have a direct extension to the quantum level. On the  $\mathcal{N} = 4$  SYM side, there are strong indications that the one- and two-loop integrability of the dilatation operator extends to all loop orders [45, 46, 69].

It is important to understand if the precise check of the string-theory / gauge-theory correspondence in the large spin sector of states may be extended to other semiclassical string states with large oscillation numbers. An indication that this is indeed the case comes from recent work [21]. As was noticed earlier in [39], the circular string oscillating in  $S^5$  (but not in  $AdS_5$ ) has energy that admits a regular expansion in  $\frac{\lambda}{N^2}$ , where  $N$  is the oscillation level number. The leading term in this expansion was matched in [16, 21] onto a particular eigenvalue of the corresponding [22]  $SO(6)$  spin chain Hamiltonian.

This raises the question of generalization of the rotation ansatz (4.2),(6.1) of the previous sections to include the possibility of string oscillations, i.e. of changing of string shape in time. It is not a priori clear which should be the most general

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\*Since the classical energy of multi-spin (say  $(J_1, J_2)$ ) solutions reproduces the near-BMN spectrum in the limit  $J_1 \gg J_2$  (see (7.36)), computing the 1-loop sigma model correction to the energy would effectively determine the  $1/J$  (2-loop, in BMN case) correction to the BMN spectrum and thus could be compared to the result of [40, 41].

rotation/oscillation ansatz for the  $\sigma$  and  $\tau$  dependence of the  $AdS_5 \times S^5$  coordinates consistent with the full 2-d classical sigma model equations of motion, but for each consistent ansatz one should expect that the 2-d sigma model should again reduce to an integrable 1-d system, whose solutions (and thus their energies) could be found in a relatively explicit way. <sup>†</sup>

An example is provided by a “2d-dual” version of the rotation ansatz (4.2) with  $\tau$  and  $\sigma$  interchanged (but keeping the  $AdS_5$  time as  $t = \kappa\tau$ ), i.e. [35]

$$X_i = z_i(\tau) e^{im_i\sigma} = r_i(\tau) e^{i\alpha_i(\tau) + im_i\sigma}, \quad \sum_{i=1}^3 r_i^2(\tau) = 1. \quad (8.1)$$

In this case the radial directions depend on  $\tau$  instead of  $\sigma$  and the “frequencies”  $m_i$  must take integer values in order to satisfy the closed string periodicity condition. This ansatz describes an “oscillating” or “pulsating”  $S^5$  string configurations, special cases of which (with motion in both  $AdS_5$  and  $S^5$ ) were discussed previously in [36, 8, 39, 71, 21]. Since the sigma model Lagrangian (2.6) is formally invariant under  $\sigma \leftrightarrow \tau$ , the resulting 1-d effective Lagrangian will have essentially the same form as (4.11),(4.15)

$$L = \frac{1}{2} \sum_{i=1}^3 (\dot{z}_i \dot{z}_i^* - m_i^2 z_i \dot{z}_i^*) + \frac{1}{2} \Lambda \left( \sum_{i=1}^3 z_i \dot{z}_i^* - 1 \right). \quad (8.2)$$

Solving for  $\dot{\alpha}_i$  as in (4.16) we get  $r_i^2 \dot{\alpha}_i = \mathcal{J}_i = \text{const}$ , where the counterparts of the integration constants  $v_i$  are now the angular momenta in (4.7). Then we end up with the following analogue of (4.17)

$$L = \frac{1}{2} \sum_{i=1}^3 \left( \dot{r}_i^2 - m_i^2 r_i^2 - \frac{\mathcal{J}_i^2}{r_i^2} \right) + \frac{1}{2} \Lambda \left( \sum_{i=1}^3 r_i^2 - 1 \right). \quad (8.3)$$

Thus pulsating solutions (carrying also 3 spins  $\mathcal{J}_i$ ) are again described by a special Neumann-Rosochatius integrable system [35]. Since the corresponding conformal gauge constraints are also  $\tau \leftrightarrow \sigma$  symmetric, they take the form similar to (2.7),(2.8) or (4.19),(4.20):  $\kappa^2 = \sum_{i=1}^3 (r_i^2 + m_i^2 r_i^2 + \frac{\mathcal{J}_i^2}{r_i^2})$  and  $\sum_{i=1}^3 m_i \mathcal{J}_i = 0$ . One may then look for periodic solutions of the above NR system (8.3) subject to the above constraints constraint, i.e. having finite 1-d energy. The resulting class of pulsating string solutions deserves a detailed study. In the simplest (“elliptic”) case reducing to a sine-Gordon type system we may follow [72, 36, 39] and introduce, as for any periodic solitonic solution, an oscillation “level number”  $N$ . In the case of the  $S^5$  pulsating solution in [39, 21] the expansion of the energy at large level  $N \gg 1$  appears to be regular in  $\frac{\lambda}{N^2}$  and, moreover, the leading  $\frac{\lambda}{N}$  term in  $E$  can be matched onto a particular anomalous dimension on the SYM side [16, 21].

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<sup>†</sup>An alternative to direct procedure of finding classical solutions may be the semiclassical quantization method used in [36, 39, 21].

One would certainly like to go beyond comparison of particular string states to particular SYM operators and to establish a more general relation between the string sigma model and the dilatation operator on the SYM side, implied by the emergence of similar integrable structures on the two sides [20, 21]. The spin chain Hamiltonian may be associated (in a thermodynamic limit) to an effective coset sigma model with the same global symmetries. Then one may hope to relate this sigma model to ( $J \rightarrow \infty$  limit of) the string sigma model by a kind of duality transformation (for a recent progress see [77]).<sup>‡</sup>

Finally, one would like also to extend the successes of checking the gauge/string duality in the non-supersymmetric semiclassical sectors of states from  $\mathcal{N} = 4$  SYM theory to less supersymmetric gauge theories. As was already mentioned in the introduction, evidence of integrable structures in the high-energy (near-conformal) limit of QCD appeared in [24, 25, 26, 27, 28], and so the spin chain relation of the 1-loop dilatation operator of  $\mathcal{N} = 4$  SYM theory [22, 23, 33] should have generalizations to other  $\mathcal{N} = 1, 2$  supersymmetric theories (and not only in twist 2 sector [73]). On the string side, while finding similar classical rotating string solutions in other less supersymmetric conformal  $AdS_5 \times M^5$  (and non-conformal, see, e.g., [37, 74]) backgrounds in type IIB theory or its orbifolds is, in principle, straightforward, it is not clear if the string sigma model corrections to the leading terms in the classical energy are again suppressed in the  $J \rightarrow \infty$  limit. For example, in the single  $S^5$  spin point-like string (BMN) case in type 0 string theory setting [75] there is a non-trivial 1-loop string correction to the energy of the twisted-sector states (which is non-analytic in  $\frac{\lambda}{J^2}$  but going to zero when  $\frac{\lambda}{J^2} \rightarrow 0$ ) [76]. Similar corrections are expected also for extended spinning string solutions, complicating direct comparison to perturbative gauge theory results.

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<sup>‡</sup>Note also that there is a known relation between a discrete version of the Neumann model and spin chains [32], but discretising  $\sigma$  in the string sigma model seems to imply a departure from the planar limit on the dual SYM side.



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